

$O(-2)$ Blow-up Formula via Instanton Calculus on $\widehat{\mathbf{C}^2/\mathbf{Z}_2}$ and Weil Conjecture

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Abstract

We calculate Betti numbers of the framed moduli space of instantons on $\widehat{\mathbf{C}^2/\mathbf{Z}_2}$, under the assumption that the corresponding torsion free sheaves E have vanishing properties ($\text{Hom}(E, E(-l_\infty)) = \text{Ext}^2(E, E(-l_\infty)) = 0$). Moreover we derive the generating function of Betti numbers and obtain closed formulas. On the other hand, we derive a universal relation between the generating function of Betti numbers of the moduli spaces of stable sheaves on X with an A_1 -singularity and that on \hat{X} blow-uped at the singularity, by using Weil conjecture. We call this the $O(-2)$ blow-up formula. Applying this to $X = \mathbf{C}^2/\mathbf{Z}_2$ case, we reproduce the formula given by instanton calculus.

1 Introduction

In the recent years, instanton calculus has been used to determine the non-perturbative effects of supersymmetric theories exactly, which are interpreted as prepotentials or the partition functions [29, 30, 7, 32]. Key point of these successes is owing to localization theorem. We apply this instanton calculus to determine the twisted $\mathcal{N} = 4$ $SU(2)$ partition function on $\widehat{\mathbf{C}^2/\mathbf{Z}_2}$ and the generating function of Poincaré polynomials of instanton moduli spaces for rank two. These partition function and generating function of Poincaré polynomials are written by beautiful closed formulas as follows. For the framed moduli spaces of instantons with rank 2 and Chern classes $c_1 = 0, c_2 = n$ on $\widehat{\mathbf{C}^2/\mathbf{Z}_2}$, the generating function of Poincaré polynomials of them is given by

Theorem 1

$$\begin{aligned} & \sum_{n \in \frac{\mathbf{Z}}{2}}^{\infty} P_t(\hat{M}(2, 0, n)) q^n \\ &= \prod_{d=1}^{\infty} \frac{1}{(1 - t^{4d} q^d)(1 - t^{4d-2} q^d)^2(1 - t^{4d-4} q^d)} \prod_{d=1}^{\infty} \frac{1 - (-t^2 q^{\frac{1}{2}})^d t^{-2}}{1 + (-t^2 q^{\frac{1}{2}})^d}. \end{aligned} \quad (1.1)$$

On the other hand, we derive the above formula in the different way. Let X be a complex surface with an A_1 -singularity, and \hat{X} be the surface blow-uped at the singularity. By using Weil conjecture and elementary transformations, we derive a universal relation between the generating function of Betti numbers of the moduli spaces of stable sheaves on X and that on \hat{X} , called as the $\mathcal{O}(-2)$ blow-up formula. After deriving the generating function of Betti numbers of the moduli spaces of stable sheaves on $\mathbf{C}^2/\mathbf{Z}_2$, we reproduce (1.1) by using the $\mathcal{O}(-2)$ blow-up formula.

In [12, 13], we used the $\mathcal{O}(-2)$ blow-up formula in order to verify the equivalence between the twisted $\mathcal{N} = 4$ partition function on orbifold T^4/\mathbf{Z}_2 and that on $K3$. It is well-known that $K3$ surface is constructed by orbifold T^4/\mathbf{Z}_2 [6]. First one divides T^4 surface by \mathbf{Z}_2 . Then 16 singularities appear. So one makes a smooth surface from this singular surface by blowing up these 16 singularities. The resulting smooth surface is Kummer surface (a kind of $K3$ surface). We tried to reconstruct the geometrical processes in the partition function level [12, 13]. In the above processes, the singular surface T^4/\mathbf{Z}_2 (the simply divided surface by \mathbf{Z}_2) is denoted by S_0 and called the contribution from S_0 as that from the untwisted sector. Blowing up a singularity, we obtain the contribution from the blow-up process, and call them a blow-up formula. When we blow up 16 singularities in S_0 , we obtain 16th power of the blow-up formula and call them as the contribution from the twisted sector. We combine the untwisted sector with the twisted sector, so that the total partition function reproduces the modular property of $\mathcal{N} = 4$ on $K3$ [21]. Finally we obtain the same partition function as that on $K3$ given by Vafa and Witten [39]. However the blow-up formula used above is not rigorous mathematically. There are two evidences of the justification. First, the $\mathcal{O}(-1)$ blow-up formula, which was introduced by Yoshioka et al. [41, 18, 19] has the form $\theta_{A_{n-1}}^{(1)}(\tau)/\eta(\tau)^n$ for $SU(n)$. Here $\theta_{A_{n-1}}^{(1)}(\tau)$ and $\eta(\tau)$ are a level 1 A_{n-1} theta function and Dedekind's eta function respectively (see Appendix A). In [41], Yoshioka calculated the effect of the blowing up process:

$\hat{\mathbf{P}}^2 \rightarrow \mathbf{P}^2$ to the partition function. Here $\hat{\mathbf{P}}^2$ stands for one point blowing up at the origin. The $\mathcal{O}(-1)$ blow-up formula is given by the ratio of the partition function on $\hat{\mathbf{P}}^2$ and that on \mathbf{P}^2 . For the $\mathcal{O}(-2)$ blow-up formula, we replace $\theta_{A_{n-1}}^{(1)}(\tau)$ by $\theta_{A_{n-1}}^{(2)}(\tau)$, since we use an $\mathcal{O}(-2)$ curve in blowing up a singularity in S_0 . Secondary, we have another evidence from a stringy picture. The $\mathcal{N} = 4$ $SU(n)$ super Yang-Mills theory on $K3$ can be derived from a IIA string compactified on $T^2 \times K3 \times ALE_{A_{n-1}}$ [14]. Here $ALE_{A_{n-1}} = \widehat{\mathbf{C}^2/\mathbf{Z}_n}$. On the other hand, the $\mathcal{N} = 4$ $U(1)^{24}$ super Yang-Mills theory on $ALE_{A_{n-1}}$ can be derived from a heterotic string compactified on $T^2 \times T^4 \times ALE_{A_{n-1}}$. These two $\mathcal{N} = 4$ theories are connected by hetero-IIA string duality[9]. The $\mathcal{N} = 4$ $U(1)$ partition function on $ALE_{A_{n-1}}$ was already determined by Nakajima[24]. It has the same form as a blow-up formula as expected. The twisted $\mathcal{N} = 4$ partition function is given by the generating function of Euler numbers of instanton moduli spaces [39, 17], which can be considered as a special case of the generating function of Poincaré polynomials with parameter $t = -1$. Thus, if we can calculate the generating function of Poincaré polynomials of instanton moduli spaces, we easily obtain the twisted $\mathcal{N} = 4$ partition function. So, we concentrate on deriving the generating function for the $\mathcal{O}(-2)$ blow-up formula.

Here we briefly explain the blow-up formula for the S -duality conjecture of Vafa-Witten[39]. We do not consider the conventional blow-up formula given by Fintushel and Stern[5] in this article. Roughly speaking, the blow-up formula is the universal relation between the Euler numbers(or the generating function of the Betti numbers) of instanton moduli spaces on a smooth four manifold and those on the blow-up of the smooth four manifold. The universal relation is independent of the four manifold. Yoshioka derived this formula by using Weil conjecture and elementary transformations[41]. Li and Qin also derived this formula by using virtual Hodge polynomials[3] in the same way[18, 19]. Kapranov derived the similar formula corresponding to an $\mathcal{O}(-d)$ blow-up formula in the connection with Kac-Moody algebra[16].

Nakajima and Yoshioka studied the framed moduli space of instantons on $\hat{\mathbf{C}}^2$ by using the technique of instanton calculus [24, 26, 27]. In [24], they reproduce the $\mathcal{O}(-1)$ blow-up formula derived in the different way[41, 18, 19]. Let F_2 be a Hirzebruch surface which has an $\mathcal{O}(-2)$ curve[1]. Considering a framed moduli space of torsion free sheaves E on F_2 , we derive Betti numbers of the framed moduli spaces of instantons on $\widehat{\mathbf{C}^2/\mathbf{Z}_2}$. Moreover we obtain the generation function of them as a beautiful closed formula (1.1). This formula coincides with the result given by T.Hausel[11]. In the calculation, we assume vanishing properties of the torsion free sheaves ($Hom(E, E(-l_\infty)) = Ext^2(E, E(-l_\infty)) = 0$). We introduce fractional line bundles which are not defined rigorously, since we want to see the relation between level 2 theta functions and the $\mathcal{O}(-2)$ curve blow-up formula(See also Sec.A.2). To derive the $\mathcal{O}(-2)$ blow-up formula, we have to determine Betti numbers of the moduli spaces of instantons on $\mathbf{C}^2/\mathbf{Z}_2$. However we cannot determine this by using the technique of instanton calculus. Thus we do in the different way.

Yoshioka derived the $\mathcal{O}(-1)$ blow-up formula by using Weil conjecture and elementary transformations[41]. We want to generalize his methods to the $\mathcal{O}(-2)$ case. The most difficult point is the precise definition of the stable vector bundle on a singular surface. However we avoid this difficult point and consider the formal generalization of the $\mathcal{O}(-1)$ case to the $\mathcal{O}(-2)$ case. Here we think that

the Weil conjecture is valid on a singular surface with an A_1 -singularity and use this conjecture on the surface. Under the assumption of the existence of the stable vector bundles, we obtain a universal relation between the generating function of Betti numbers of the moduli spaces of stable sheaves on X with an A_1 -singularity and that on \hat{X} blow-uped at the singularity. We check this $\mathcal{O}(-2)$ blow-up formula for $X = \mathbf{C}^2/\mathbf{Z}_2$ case, and obtain the same result as one given by instanton calculus. The generating function of Betti numbers of the moduli spaces of stable sheaves on $\mathbf{C}^2/\mathbf{Z}_2$ is obtained as follows. We separate the contribution on $\widehat{\mathbf{C}^2/\mathbf{Z}_2}$ into that from the untwisted sector and that from the twisted sector. Then, the contribution from the untwisted sector can be obtained by taking \mathbf{Z}_2 -invariant part of the contribution on \mathbf{C}^2 .

The organization of the article is as follows. In Sec.2, we introduce the blow-up and Weil conjecture used below. In Sec.3.1, we introduce the framed moduli space of torsion free sheaves on $\mathbf{P}^2(F_2)$ and that of instanton on $S^4(\widehat{\mathbf{C}^2/\mathbf{Z}_2})$. In Sec.3.2, we study torus actions and their fixed point set. In Sec.3.3, we calculate Betti numbers of the framed moduli space of torsion free sheaves on $\mathbf{P}^2(F_2)$. In Sec.4.1, we calculate Betti numbers of the moduli space of stable vector bundles on $\hat{X}(X)$. In Sec.4.2, we provide the useful formulas to count Betti numbers of the moduli spaces of stable sheaves of non-vector bundle on $\hat{X}(X)$. Then, we derive the $\mathcal{O}(-2)$ blow-up formula. In Sec.4.3, we derive Betti numbers of the moduli spaces of stable sheaves on $\mathbf{C}^2/\mathbf{Z}_2$. Then, we reproduce the formula given in Sec.3.3 by using the $\mathcal{O}(-2)$ blow-up formula. In Sec.5, we summarize our results and make some comments on the other related works. In Sec.A.1, we introduce level l theta functions used in this article. In Sec.A.2, we show that level 2 theta functions appear naturally in instanton calculus on $\widehat{\mathbf{C}^2/\mathbf{Z}_2}$. In Sec.B, we verify an identity appeared in derivation of (1.1). In Sec.C, we calculate Betti numbers of the framed moduli spaces for odd c_1 under some assumptions.

2 Preliminary

2.1 Blow-up

Following [6], we introduce the blow-up:

Definition 1 *Under the immersion*

$$I : \mathbf{C}^2 - \{\vec{0}\} \rightarrow \mathbf{C}^2 \times \mathbf{P}^1, \quad (2.1)$$

the closure of the image is denoted by $\hat{\mathbf{C}}^2$. We call $\hat{\mathbf{C}}^2$ as the blow-up of \mathbf{C}^2 at $\{\vec{0}\}$.

We introduce an open subset of \mathbf{C}^2 as

$$D_C^2 = \{(z_1, z_2) \in \mathbf{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\}. \quad (2.2)$$

The whole complex linear maps preserving inner product are denoted by $U(2)$ and the finite subgroup is denoted by Γ . We consider the immersion

$$I : D_C^2 - \{\vec{0}\} \rightarrow D_C^2 \times \mathbf{P}^1. \quad (2.3)$$

Then Γ -action is preserved. So, we can also define the following blow-up:

Definition 2

$$\bar{I} : (D_C^2 - \{\vec{0}\})/\Gamma \rightarrow D_C^2 \times \mathbf{P}^1/\Gamma, \quad (2.4)$$

the closure of the image is denoted by $\widehat{D_C^2}/\Gamma$. We call $\widehat{D_C^2}/\Gamma$ as the blow-up of D_C^2/Γ at $\{\vec{0}\}$.

In $\Gamma = \mathbf{Z}_r$ case, we consider that D_C^2/\mathbf{Z}_{r+1} has an A_r -singularity.

2.2 Weil Conjectures

Let \mathbf{F}_q be a finite field with q elements, $\bar{\mathbf{F}}_q$ the algebraic closure of \mathbf{F}_q . For a scheme X over \mathbf{F}_q , \bar{X} denotes $X \otimes_{\mathbf{F}_q} \bar{\mathbf{F}}_q$ and $X(\mathbf{F}_q)$ the set of \mathbf{F}_q -rational points of X . For a smooth projective variety X over \mathbf{C} , $b_i(X)$ is the i -th Betti number of X , $\chi(X) := \sum_i (-1)^i b_i(X)$ the Euler number of X , and

$$P_t(X) := \sum_i b_i(X) t^i \quad (2.5)$$

the Poincaré polynomial of X . The zeta function of X over \mathbf{F}_q is defined as

$$Z_q(X, t) := \exp\left(\sum_{r>0} (\#X(\mathbf{F}_{q^r}) \frac{t^r}{r})\right). \quad (2.6)$$

Theorem 2 (Deline) [4]

Let X be a smooth projective variety of dimension n over \mathbf{F}_q .

(1) $Z_q(X, t)$ is a rational function on t .

(2)

$$Z_q(X, t) = \prod_{i=0}^{2n} P_i(X, t)^{i+1}, \quad (2.7)$$

where $P_i(X, t) = \prod_{j=1}^{b_i} (1 - \alpha_{i,j} t)$, $|\alpha_{i,j}| = q^{i/2}$.

(3) We have

$$Z_q(X, 1/q^n t) = \pm q^{e(\bar{X})/2} t^{e(\bar{X})} Z_q(X, t), \quad (2.8)$$

where $e(\bar{X}) = \sum_i (-1)^i \deg P_i$.

(4) If X is a good reduction of a smooth projective variety Y over \mathbf{C} , then $b_i = b_i(Y)$.

From this theorem, replacing $\alpha_{i,j}$ by $(-z)^i$, we obtain the Poincaré polynomial of Y :

$$\#X(\mathbf{F}_q) = \sum_{i,j} (-1)^i \alpha_{i,j} \rightarrow P_z(Y) = \sum_i b_i(Y) z^i. \quad (2.9)$$

We consider the case of $X = \mathbf{P}^n$ for an example. By using the Frobenius morphism $F : X_{\bar{\mathbf{F}}_q} \rightarrow X_{\bar{\mathbf{F}}_q}$ sending $(z_0 : z_1 : \cdots : z_n)$ to $(z_0^q : z_1^q : \cdots : z_n^q)$, we obtain the fixed point set $\{x \in X_{\bar{\mathbf{F}}_q} | F(x) = x\} = X(\mathbf{F}_q)$. By Lefschetz

fixed point theorem, $\#X(\mathbf{F}_q) = \sum_i (-1)^i \text{tr}(F_i^*) = \sum_i^n q^i$, where $F_i^* : H^i(X) \rightarrow H^i(X)$ is the endomorphism induced by F . Thus we obtain

$$\#\mathbf{P}^n(\mathbf{F}_q) = \sum_{i=0}^n q^i \xrightarrow{\sim} P_z(\mathbf{P}^n) = \sum_{i=0}^n z^{2i}, \quad (2.10)$$

where we set $q = z^2$.

Although the Weil conjecture is true on a smooth projective variety, we apply this to the case of X = a singular surface with an A_r -singularity in this article.

3 Instanton Calculus on $\widehat{\mathbf{C}^2/\mathbf{Z}_2}$

3.1 Framed Moduli Space

The framed moduli space is mainly defined in the two ways as that of instanton and that of torsion free sheaf. At first, we think of the framed moduli space of instantons on S^4 and that of torsion free sheaves on \mathbf{P}^2 [25].

We introduce the hyper-Kähler geometry used below [23].

Definition 3 *Let X be a $4n$ -dimensional manifold. A hyper-Kähler structure of X consists of a Riemannian metric g and a triple of almost complex structures $\mathcal{I}, \mathcal{J}, \mathcal{K}$ which satisfy the following conditions:*

(1)

$$g(\mathcal{I}v, \mathcal{I}w) = g(\mathcal{J}v, \mathcal{J}w) = g(\mathcal{K}v, \mathcal{K}w) = g(v, w) \text{ for } v, w \in TX. \quad (3.1)$$

(2) $(\mathcal{I}, \mathcal{J}, \mathcal{K})$ satisfies a relation

$$\mathcal{I}^2 = \mathcal{J}^2 = \mathcal{K}^2 = \mathcal{I}\mathcal{J}\mathcal{K} = -1. \quad (3.2)$$

(3) \mathcal{I}, \mathcal{J} and \mathcal{K} are parallel with respect to the Levi-Civita connection of g ,

$$\nabla \mathcal{I} = \nabla \mathcal{J} = \nabla \mathcal{K} = 0. \quad (3.3)$$

(4) For \mathcal{I}, \mathcal{J} and \mathcal{K} , 2-forms ω_1, ω_2 and ω_3 are defined by

$$\omega_1(v, w) := g(\mathcal{I}v, w), \omega_2(v, w) := g(\mathcal{J}v, w), \omega_3(v, w) := g(\mathcal{K}v, w) \text{ for } v, w \in TX, \quad (3.4)$$

which satisfy $d\omega_1 = d\omega_2 = d\omega_3 = 0$. (These ω_1, ω_2 and ω_3 are called the Kähler forms associated with (g, \mathcal{I}) , (g, \mathcal{J}) and (g, \mathcal{K}) , respectively.)

Suppose that a compact Lie group G acts on X preserving $g, \mathcal{I}, \mathcal{J}, \mathcal{K}$. The Lie algebra of G is denoted by \mathcal{G} , and its dual is denoted by \mathcal{G}^* .

Definition 4 *A map*

$$\mu = (\mu_1, \mu_2, \mu_3) : X \rightarrow \mathbf{R}^3 \otimes \mathcal{G}^* \quad (3.5)$$

is said to be a hyper-Kähler momentum map if it satisfies the following conditions:

(1) μ is G -equivalent, i.e. $\mu(g \cdot x) = \text{Ad}_{g^{-1}}^* \mu(x)$.

(2) $\langle d\mu_i(v), \xi \rangle = \omega_i(\xi^*, v)$ for any $v \in TX$, any $A \in \mathcal{G}$ and $i = 1, 2, 3$, where ξ^* is a vector field generated by ξ .

Let V, W be hermitian vector spaces whose dimensions are n, r , respectively. For these spaces, we define a complex vector space $\mathbf{M}(r, n)$ as

$$\mathbf{M}(r, n) := \{(B_1, B_2, I, J) | B_1, B_2 \in \text{Hom}(V, V), I \in \text{Hom}(W, V), J \in \text{Hom}(V, W)\}. \quad (3.6)$$

We consider an action of $g \in U(V)$ on $\mathbf{M}(r, n)$ given by

$$\mathbf{M}(r, n) \ni (B_1, B_2, I, J) \rightarrow (g^{-1}B_1g, g^{-1}B_2g, g^{-1}I, Jg). \quad (3.7)$$

We define a momentum map $\mu_1 : \mathbf{M}(r, n) \rightarrow U(V)$,

$$\mu_1(B_1, B_2, I, J) := \frac{i}{2}([B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J). \quad (3.8)$$

We also define a momentum map $\mu_{\mathbf{C}} : \mathbf{M}(r, n) \rightarrow \text{End}(V)$,

$$\mu_{\mathbf{C}}(B_1, B_2, I, J) := [B_1, B_2] + IJ. \quad (3.9)$$

Note that $\mu_{\mathbf{C}} = \mu_2 + i\mu_3$ in this case.

Definition 5 *The framed moduli space of instantons on S^4 with rank r and second Chern class n is defined by*

$$M(r, n) := \mu_1^{-1}(\frac{i}{2}\zeta \text{id}) \cap \mu_{\mathbf{C}}^{-1}(0)/U(V), \quad (3.10)$$

where ζ is a fixed positive real number.

This space is known to be non-singular of dimension $2nr$. $\mu_1^{-1}(\cdots) \cap \mu_{\mathbf{C}}^{-1}(\cdots)/U(V)$ stands for a hyper-Kähler quotient. We also give another type of description of the framed moduli space of instanton on S^4 :

Lemma 1 [23] *The framed moduli space of instantons on S^4 with rank r and second Chern class n is given by*

$$M(r, n) = \{(B_1, B_2, I, J) \in \mu_{\mathbf{C}}^{-1}(0) | (B_1, B_2, I, J) \text{ is stable}\} / U(V), \quad (3.11)$$

where the stable (B_1, B_2, I, J) has no subspace $S \subset V$ which satisfies

$$B_\alpha(S) \subset S \text{ and } \text{Im}(I) \subset S. \quad (3.12)$$

We define the framed moduli space of ideal instantons on S^4 with rank r and second Chern class n :

Definition 6 *The framed moduli space of ideal instanton on S^4 with rank r and second Chern class n is defined by*

$$M_0(r, n) := \mu_1^{-1}(0) \cap \mu_{\mathbf{C}}^{-1}(0)/U(V). \quad (3.13)$$

We also give the framed moduli space of ideal instantons on S^4 differently:

Lemma 2 [23] *The framed moduli space of ideal instantons on S^4 with rank r and second Chern class n is given by*

$$\begin{aligned} M_0(r, n) &= \mu_{\mathbf{C}}^{-1}(0) // GL(V) \\ &= \text{the set of closed } GL(V)\text{-orbits in } \mu_{\mathbf{C}}^{-1}(0), \end{aligned} \quad (3.14)$$

where $//$ means the affine algebro-geometric quotient.

Since this space has singularities, we take the non-singular locus defined by

Definition 7

$$M_0^{reg}(r, n) := \{[(B_1, B_2, I, J)] \in M_0(r, n) | \text{the stabilizer in } U(V) \text{ of } (B_1, B_2, I, J) \text{ is trivial}\}. \quad (3.15)$$

This space is identified with the moduli space of genuine instantons on S^4 . $M_0^{reg}(r, n)$ and $M_0(r, n)$ is related by

$$M_0(r, n) = \coprod_{k=0}^n M_0^{reg}(r, n-k) \times S^k \mathbf{C}^2. \quad (3.16)$$

This means that $M_0(r, n)$ is an Uhlenbeck compactification of M_0^{reg} . Here $S^n X$ stands for n -th symmetric product of X .

Here we mention the difference between the framed moduli space of instantons and the moduli space of instantons by using analytic terms[23]. First we show that the moduli space of instantons on a 4-dimensional hyper-Kähler manifold X is a hyper-Kähler quotient. For a smooth vector bundle E over X with a Hermitian metric and the space of metric connections \mathcal{A} on E , the tangent space $T_A \mathcal{A}$ at $A \in \mathcal{A}$ is identified with $T_A \mathcal{A} \cong \Omega^1(\mathcal{U}(E))$. For the tangent space $T_A \mathcal{A}$, we have a natural L^2 -metric and almost complex structures on $T_A \mathcal{A}$ induced from those on X . We also have the group of gauge transformations \mathcal{G} acting on \mathcal{A} . Then, the hyper-Kähler momentum map of the action of \mathcal{G} on \mathcal{A}

$$\mu = (\mu_1, \mu_2, \mu_3) : \mathcal{A} \rightarrow \mathbf{R}^3 \otimes Lie \mathcal{G}^* \cong \mathbf{R}^3 \otimes \Omega^4(\mathcal{U}(E)) \quad (3.17)$$

is given by

$$\mu_i(\mathcal{A}) = F_A^+ \wedge \omega_i \in \Omega^4(\mathcal{U}(E)) \quad (i = 1, 2, 3). \quad (3.18)$$

Here F_A^+ is a self-dual part of the curvature 2-form of A , and ω_i is the Kähler form associated with the complex structures on X . We define the moduli space of instantons on X by $\mu^{-1}(0)/\mathcal{G}$. This construction works even in the case $X = \mathbf{C}^2$. The framed moduli space of instantons is considered as a quotient by a group of gauge transformations converging to the identity at the end of X . Thus in the case $X = \mathbf{C}^2$ we consider one point compactification $S^4 = \mathbf{C}^2 \cup \{\infty\}$. The difference between the framed moduli space of instantons and the moduli space of instantons is whether the corresponding vector bundles have a converging property for a group of gauge transformations or not.

To mention the relation between $M(r, n)$ and $M_0(r, n)$, we define the framed moduli space of torsion free sheaves on \mathbf{P}^2 , which is an alternative definition of the framed moduli space.

Let $M(r, n)$ be the framed moduli space of torsion free sheaves on \mathbf{P}^2 with rank r and $c_2 = n$, which parametrizes isomorphism classes of (E, Φ) such that

- (1) E is a torsion free sheaf of $\text{rank}(E) = r, < c_2(E), [\mathbf{P}^2] \geq n$ which is locally free in a neighborhood of l_∞ ,
- (2) $\Phi : E|_{l_\infty} \rightarrow \mathcal{O}_{l_\infty}^{\oplus r}$ is an isomorphism called ‘framing at infinity’.

Here $l_\infty = \{[0 : z_1 : z_2] \in \mathbf{P}^2\} \subset \mathbf{P}^2$ is the line at infinity. Note that the existence of a framing Φ implies $c_1(E) = 0$. The equivalence between the framed moduli spaces of instanton on S^4 and that of torsion free sheaves on \mathbf{P}^2 is explained in detail[23]. Using this definition, we denote the relation between $M(r, n)$ and $M_0(r, n)$

Theorem 3 [25] *There is a projective morphism*

$$\pi : M(r, n) \rightarrow M_0(r, n) \quad (3.19)$$

defined by

$$(E, \Phi) \rightarrow (E^{\vee\vee}, \Phi), \text{Supp}(E^{\vee\vee}/E) \in M_0^{\text{reg}}(r, n') \times S^{n-n'} \mathbf{C}^2, \quad (3.20)$$

where $E^{\vee\vee}$ is the double dual of E and $\text{Supp}(E^{\vee\vee}/E)$ is the support of $(E^{\vee\vee}/E)$ counted with multiplicities.

This morphism π is the Hilbert-Chow morphism. As an example of the Hilbert-Chow morphism, we write a theorem:

Theorem 4 [23]

$$M(1, n) = (\mathbf{C}^2)^{[n]}, M_0(1, n) = S^n(\mathbf{C}^2), \quad (3.21)$$

where $(X)^{[n]}$ stands for the Hilbert scheme of n points on surface X .

Now we move to the framed moduli space of instanton on $\widehat{\mathbf{C}^2/\mathbf{Z}_2}$. The similar treatment is done in [25]. To describe the corresponding torsion free sheaves, we introduce the following manifold F_2 :

$$F_2 = \{([z_0 : z_1 : z_2], [z : w]) \in \mathbf{P}^2 \times \mathbf{P}^1; z_1 w^2 = z_2 z^2\} \quad (3.22)$$

as a Hirzebruch surface[1]. Let $p : F_2 \rightarrow \mathbf{P}^2$ denote the projection to the first factor. We denote the inverse image of $\{z_0 = 0\} \subset \mathbf{P}^2$ under $p : F_2 \rightarrow \mathbf{P}^2$ by l_∞ and denote the exceptional set $\{z_1 = z_2 = 0\}$ by C . Note that $F_2 \setminus l_\infty = \widehat{\mathbf{C}^2/\mathbf{Z}_2}$ and the self-intersection $[C]^2 = -2$. We interpret that $p : F_2 \rightarrow \mathbf{P}^2$ is a kind of blow-up, but is not different from that in Sec.2. The blow-up in Sec.2 is treated in the next section.

In the remaining part, \mathcal{O} denotes the structure sheaf of F_2 , $\mathcal{O}(C)$ the line bundle associated with the divisor C , $\mathcal{O}(mC)$ its m th tensor product.

Let $\hat{M}(r, k, n)$ be the framed moduli space of torsion free sheaves (E, Φ) on F_2 with $\text{rank } 2, < c_1(E), [C] \geq -k$ and $< c_2(E) - \frac{r-1}{2r} c_1(E)^2, [F_2] \geq n$. This space is non-singular of dimension $2nr$.

In the same way as the framed moduli space of torsion free sheaves on $\hat{\mathbf{P}}^2$ [25], we assume

Conjecture 1 *There is a projective morphism*

$$\hat{\pi} : \hat{M}(r, k, n) \rightarrow M_0(r, n - \frac{1}{4r} k(r - k)) (0 \leq k < 2r) \quad (3.23)$$

defined by

$$(E, \Phi) \rightarrow ((p_* E)^{\vee\vee}, \Phi), \text{Supp}(p_* E^{\vee\vee}/p_* E) + \text{Supp}(R^1 p_* E). \quad (3.24)$$

We will give an evidence of Conjecture 1 in Sec.3.3.

3.2 Torus Actions and their Fixed Point Set

Let us define an action of the $(r+2)$ -dimensional torus $\tilde{T} = \mathbf{C}^* \times \mathbf{C}^* \times T^r$ on $M(r, n)[25, 26]$. For $(t_1, t_2) \in \mathbf{C}^* \times \mathbf{C}^*$, let F_{t_1, t_2} be an automorphism of \mathbf{P}^2 defined by

$$F_{t_1, t_2}([z_0 : z_1 : z_2]) = ([z_0 : t_1 z_1 : t_2 z_2]). \quad (3.25)$$

For $\text{diag}(e_1, \dots, e_r) \in T^r$, let G_{e_1, \dots, e_r} denotes the isomorphism of $\mathcal{O}_{l_\infty}^{\oplus r}$ given by

$$\mathcal{O}_{l_\infty}^{\oplus r} \ni (s_1, \dots, s_r) \mapsto (e_1 s_1, \dots, e_r s_r). \quad (3.26)$$

Then for $(E, \Phi) \in M(r, n)$, we define

$$(t_1, t_2, e_1, \dots, e_r) \cdot (E, \Phi) = ((F_{t_1, t_2}^{-1})^* E, \Phi'), \quad (3.27)$$

where Φ' is the composite of homomorphisms

$$(F_{t_1, t_2}^{-1})^* E|_{l_\infty} \xrightarrow{(F_{t_1, t_2}^{-1})^* \Phi} (F_{t_1, t_2}^{-1})^* \mathcal{O}_{l_\infty}^{\oplus r} \xrightarrow{\mathcal{O}_{l_\infty}^{\oplus r} \xrightarrow{G_{e_1, \dots, e_r}} \mathcal{O}_{l_\infty}^{\oplus r}} \mathcal{O}_{l_\infty}^{\oplus r}. \quad (3.28)$$

Here the middle arrow is the homomorphism given by the action of \tilde{T} . Using the matrixes (B_1, B_2, I, J) , we obtain \tilde{T} action on $M(r, n)$:

$$(B_1, B_2, I, J) \mapsto (t_1 B_1, t_2 B_2, I e^{-1}, t_1 t_2 e J), \text{ for } t_1, t_2 \in \mathbf{C}^*, e = \text{diag}(e_1, \dots, e_r) \in (\mathbf{C}^*)^r. \quad (3.29)$$

Here \tilde{T} -action preserves $\mu_{\mathbf{C}}(B_1, B_2, I, J) = 0$ and the stability condition commutes the action of $GL(V)$.

In the same way, we have a \tilde{T} -action on $M_0(r, n)$. The map $\pi : M(r, n) \rightarrow M_0(r, n)$ is equivalent.

The fixed points $M(r, n)^{\tilde{T}}$ consist of $(E, \Phi) = (I_1, \Phi_1) \oplus \dots \oplus (I_r, \Phi_r)$ such that

- (1) I_α is an ideal sheaf of 0-dimensional subscheme Z_α contained in $\mathbf{C}^2 = \mathbf{P}^2 \setminus l_\infty$ ($\mathcal{O}_{\mathbf{P}^2}/I_\alpha = \mathcal{O}_{Z_\alpha}$).
- (2) Φ_α is an isomorphism from $(I_\alpha)_{l_\infty}$ to the α th factor of $\mathcal{O}_{l_\infty}^{\oplus r}$.
- (3) I_α is fixed by the action of $\mathbf{C}^* \times \mathbf{C}^*$, coming from that on \mathbf{P}^2 .

We parametrize the fixed point set $M(r, n)^{\tilde{T}}$ by r -tuple of Young diagrams $\vec{Y} = (Y_1, \dots, Y_r)$. Y_α corresponds to the ideal I_α spanned by monomials $x^h y^k$ placed at $(h+1, k+1)$ outside Y_α . The constraint is that the total number of boxes $|\vec{Y}| := \sum_\alpha |Y_\alpha|$ is equal to n .

Let $Y = (\lambda_1 \geq \lambda_2 \geq \dots)$ be a Young diagram, where λ_h is the length of the h -th column of Y , and λ'_k is the length of the k -th row of Y . Let $l(Y)$ denote the number of columns of Y , i.e., $l(Y) = \lambda'_1$. Then, we define $l_Y(s)$ and $a_Y(s)$ as

$$l_Y(s) := \lambda_h - k, \quad a_Y(s) := \lambda'_k - h, \quad (3.30)$$

where $s = (h, k) \in (\mathbf{Z}_{>0})^2$.

On the other hand, the fixed points $M_0(r, n)^{\tilde{T}}$ consist of the single point $n[0] \in S^n \mathbf{C}^2 \subset M_0(r, n)$.

Theorem 5 [25, 32] Let (E, Φ) be a fixed point of \tilde{T} -action corresponding to $\vec{Y} = (Y_1, \dots, Y_r)$. Then the \tilde{T} -module structure of $T_{(E, \Phi)}M(r, n)$ is given by

$$T_{(E, \Phi)}M(r, n) = \sum_{\alpha, \beta}^r N_{\alpha, \beta}^{\vec{Y}}(t_1, t_2) \quad (3.31)$$

where

$$N_{\alpha, \beta}^{\vec{Y}}(t_1, t_2) = e_\beta e_\alpha^{-1} \times \left\{ \sum_{s \in Y_\alpha} (t_1^{-l_{Y_\beta}(s)} t_2^{a_{Y_\alpha}(s)+1}) + \sum_{s \in Y_\beta} (t_1^{l_{Y_\alpha}(s)+1} t_2^{-a_{Y_\alpha}(s)}) \right\}. \quad (3.32)$$

Here we denote e_α ($\alpha = 1, \dots, r$) by the one dimensional \tilde{T} -module given by

$$\tilde{T} \ni (t_1, t_2, \dots, e_r) \mapsto e_\alpha.$$

Similarly t_1, t_2 denote one-dimensional \tilde{T} -modules. Then, the representation ring $R(\tilde{T})$ is isomorphic to $\mathbf{Z}[t_1^\pm, t_2^\pm, e_1^\pm, \dots, e_r^\pm]$, where e_α^{-1} is the dual of e_α .

Now we move to the case of $\hat{M}(r, k, n)^{\tilde{T}}$. We assume that $\text{Hom}(E, E(-l_\infty)) = \text{Ext}^2(E, E(-l_\infty)) = 0$ in this article. This condition is related to the smoothness of its moduli space.

Let us define an action of the $(r+2)$ -dimensional torus $\tilde{T} = \mathbf{C}^* \times \mathbf{C}^* \times T^r$ on $\hat{M}(r, k, n)$ by modifying the action of \tilde{T} on $M(r, n)$ as follows. For $(t_1, t_2) \in \mathbf{C}^* \times \mathbf{C}^*$, let F'_{t_1, t_2} be an automorphism on F_2 defined by

$$F'_{t_1, t_2}([z_0 : z_1 : z_2], [z : w]) = ([z_0 : t_1^2 z_1 : t_2^2 z_2], [t_1 z : t_2 w]), \quad (3.33)$$

where the condition $z_1 w^2 = z_2 z^2$ is preserved. Then we define the action of \tilde{T} by replacing F_{t_1, t_2} by F'_{t_1, t_2} in (3.25). The action of the latter T^r is exactly the same as before. Under a pullback $p^* : F_2 \rightarrow \mathbf{P}^2$, an automorphism on \mathbf{P}^2 is given by

$$p^* F'_{t_1, t_2}([z_0 : z_1 : z_2]) = ([z_0 : t_1^2 z_1 : t_2^2 z_2]). \quad (3.34)$$

Then, the morphism $\hat{\pi}$ is equivalent.

Note that the fixed point set of $\mathbf{C}^* \times \mathbf{C}^*$ in $\widehat{\mathbf{C}^2/\mathbf{Z}_2} = F_2 \setminus l_\infty$ consists of two points $([1 : 0 : 0], [1 : 0])$ and $([1 : 0 : 0], [0 : 1])$, which are denoted by p_1 and p_2 respectively.

The fixed points $\hat{M}(r, n)^{\tilde{T}}$ consist of $(E, \Phi) = (I_1(k_1/2), \Phi_1) \oplus \dots \oplus (I_r(k_r/2), \Phi_r)$ such that

- (1) $I_\alpha(k_\alpha/2)$ is the tensor product $I_\alpha \otimes \mathcal{O}(k_\alpha C/2)$, where $k_\alpha \in \mathbf{Z}$ and I_α is an ideal sheaf of 0-dimensional subscheme Z_α contained in $\mathbf{C}^2/\mathbf{Z}_2 = F_2 \setminus l_\infty$.
- (2) Φ_α is an isomorphism from $(I_\alpha)_{l_\infty}$ to the α th factor of $\mathcal{O}_{l_\infty}^{\oplus r}$.
- (3) I_α is fixed by the action of $\mathbf{C}^* \times \mathbf{C}^*$, coming from that on F_2 .

$\text{Supp}(Z_\alpha) = \{p_1, p_2\} \in F_2 \setminus l_\infty$. Thus $Z_\alpha = Z_\alpha^1 \cup Z_\alpha^2$, where Z_α^1 and Z_α^2 are supported at p_1 and p_2 respectively. Alternatively $I_\alpha = I_\alpha^1 \cap I_\alpha^2$, where $\mathcal{O}/I_\alpha^k = \mathcal{O}_{Z_\alpha^k}$. If we take a coordinate system $(x, y) = (z_1/z_0, w/z)((z/w, z_2/z_0))$ around $p_1(p_2)$, then $I_\alpha^1(I_\alpha^2)$ is generated by monomials $x^k y^h$. Then $I_\alpha^1(I_\alpha^2)$ corresponds to

a Young diagram $Y_\alpha^1(Y_\alpha^2)$ as before. Therefore the fixed point set is parametrized by r -tuples of $(\vec{k}, Y^1, Y^2) = ((k_1, Y_1^1, Y_1^2), \dots, (k_r, Y_r^1, Y_r^2))$, where $k_\alpha \in \mathbf{Z}$. The constraint is

$$\sum_{\alpha} k_{\alpha} = k, \quad |\vec{Y}^1| + |\vec{Y}^2| + \frac{1}{r} \sum_{\alpha < \beta} \left| \frac{k_{\alpha}}{2} - \frac{k_{\beta}}{2} \right|^2 = n. \quad (3.35)$$

One would wonder why we introduce fractional line bundles $\mathcal{O}(k_{\alpha}C/2)$. $\mathcal{O}(k_{\alpha}C/2)$ themselves are not defined rigorously. However we are interested in the relation between affine Lie algebra and instanton calculus on $\widehat{\mathbf{C}^2/\mathbf{Z}_2}$. Indeed $\mathcal{O}(-1)$ curve case [25, 26, 27] are related to level 1 theta functions. Moreover $\mathcal{O}(-2)$ curve case is also related to level 2 theta functions as mentioned in Sec.A.2. In this case, fractional line bundles must be introduced.

Hereafter we only consider the case of $r = 2$. There are two types of $\hat{M}(2, k, n)$: $k \equiv 0 \pmod{2}$ (even type) and $k \equiv 1 \pmod{2}$ (odd type). We only consider the case of $k \equiv 0 \pmod{2}$ (even type). This is because there is a difficulty in the proof of Theorem 6 in odd case. The proof is done in the same way as [26].

Theorem 6 *Let (E, Φ) be a fixed point of \tilde{T} -action corresponding to $(\vec{k}, \vec{Y}^1, \vec{Y}^2) = ((k_1, Y_1^1, Y_1^2), (k_r, Y_r^1, Y_r^2))$. Suppose $k \equiv 0 \pmod{2}$. Then the \tilde{T} -module structure of $T_{(E, \Phi)}\hat{M}(r, k, n)$ is given by*

$$T_{(E, \Phi)}\hat{M}(r, k, n) = \sum_{\alpha, \beta}^2 \left(L_{\alpha, \beta}^{\vec{k}}(t_1, t_2) + t_1^{k_{\beta} - k_{\alpha}} N_{\alpha, \beta}^{\vec{Y}^1}(t_1^2, t_2/t_1) + t_2^{k_{\beta} - k_{\alpha}} N_{\alpha, \beta}^{\vec{Y}^2}(t_1/t_2, t_2^2) \right), \quad (3.36)$$

where

$$L_{\alpha, \beta}^{\vec{k}}(t_1, t_2) = e_{\beta} e_{\alpha}^{-1} \times \begin{cases} \sum_{\substack{i, j \geq 0, i+j \equiv 0 \pmod{2} \\ i+j \leq k_{\alpha} - k_{\beta} - 2}} t_1^{-i} t_2^{-j} & \text{if } k_{\alpha} > k_{\beta} + 1, \\ \sum_{\substack{i, j \geq 0, i+j \equiv 0 \pmod{2} \\ i+j \leq k_{\beta} - k_{\alpha} - 2}} t_1^{i+1} t_2^{j+1} & \text{if } k_{\alpha} + 1 < k_{\beta}, \\ 0 & \text{otherwise.} \end{cases} \quad (3.37)$$

Proof. According to the decomposition $E = I_1(k_1C/2) \oplus \dots \oplus I_r(k_rC/2)$, the tangent space $T_{(E, \Phi)}\hat{M}(r, k, n) = \text{Ext}^1(E, E(-l_{\infty}))$ is decomposed as

$$\text{Ext}^1(E, E(-l_{\infty})) = \bigoplus_{\alpha, \beta} \text{Ext}^1(I_{\alpha}(k_{\alpha}C/2), I_{\beta}(k_{\beta}C/2 - l_{\infty})). \quad (3.38)$$

The factor $\text{Ext}^1(I_{\alpha}(k_{\alpha}C/2), I_{\beta}(k_{\beta}C/2 - l_{\infty}))$ has weight $e_{\beta} e_{\alpha}^{-1}$ as a T -module. We only have to describe each factor as a T^2 -module.

Under the assumption that $\text{Hom}(E, E(-l_{\infty})) = \text{Ext}^2(E, E(-l_{\infty})) = 0$, $\text{Ext}^1(I_{\alpha}(k_{\alpha}C/2), I_{\beta}(k_{\beta}C/2 - l_{\infty})) = -\sum_{i=0} (-1)^i \text{Ext}^i(I_{\alpha}(k_{\alpha}C/2), I_{\beta}(k_{\beta}C/2 - l_{\infty}))$. Thus, using the exact sequence $0 \rightarrow I_{\alpha} \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{Z_{\alpha}} \rightarrow 0$, we have

$$\begin{aligned} & \sum_{i=0} (-1)^i \text{Ext}^i(I_{\alpha}(k_{\alpha}C/2), I_{\beta}(k_{\beta}C/2 - l_{\infty})) \\ &= \sum_{i=0} (-1)^i \text{Ext}^i(\mathcal{O}(k_{\alpha}C/2), \mathcal{O}(k_{\beta}C/2 - l_{\infty})) - \sum_{i=0} (-1)^i \text{Ext}^i(\mathcal{O}(k_{\alpha}C/2), \mathcal{O}_{Z_{\beta}}(k_{\beta}C/2 - l_{\infty})) \end{aligned}$$

$$-\sum_{i=0}(-1)^i \text{Ext}^i(\mathcal{O}_{Z_\alpha}(k_\alpha C/2), \mathcal{O}(k_\beta C/2 - l_\infty)) + \sum_{i=0}(-1)^i \text{Ext}^i(\mathcal{O}_{Z_\alpha}(k_\alpha C/2), \mathcal{O}_{Z_\beta}(k_\beta C/2 - l_\infty)). \quad (3.39)$$

First we consider the term $\sum_{i=0}(-1)^i \text{Ext}^i(\mathcal{O}(k_\alpha C/2), \mathcal{O}(k_\beta C/2 - l_\infty)) = -\text{Ext}^1(\mathcal{O}(k_\alpha C/2), \mathcal{O}(k_\beta C/2 - l_\infty)) = -H^1(\mathcal{O}((k_\beta - k_\alpha)C/2 - l_\infty)) = -L_{\alpha, \beta}$. We set $n = k_\alpha - k_\beta$. Since $H^1(F_2, \mathcal{O}(-l_\infty)) = 0$, we only consider two cases of $n > 1$ and $n < -1$. On the other hand, $H^1(F_2, \mathcal{O}(\pm C/2 - l_\infty)) = 0$ may not happen. This is the difficulty in odd type.

Let us consider the case of $n > 1$. We consider the cohomology long exact sequence associated with an exact sequence $0 \rightarrow \mathcal{O}(-nC/2) \rightarrow \mathcal{O}((-n+2)C/2) \rightarrow \mathcal{O}_C((-n+2)C/2) \rightarrow 0$, which is equivalent under $\mathbf{C}^* \times \mathbf{C}^*$ -action. Since C is a projective line with self-intersection (-2) , we have $H^1(C, \mathcal{O}_C((-n+2)C/2)) = H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(n-2)) = 0$. Thus we have

$$0 \rightarrow H^0(C, \mathcal{O}_{\mathbf{P}^1}(n-2)) \rightarrow H^1(F_2, \mathcal{O}((-n)C/2 - l_\infty)) \rightarrow H^1(F_2, \mathcal{O}((-n+2)C/2 - l_\infty)) \rightarrow 0, \quad (3.40)$$

which is an exact sequence in $\mathbf{C}^* \times \mathbf{C}^*$ -modules. Starting with $H^1(F_2, \mathcal{O}(-l_\infty)) = 0$, we obtain

$$H^1(F_2, \mathcal{O}(-nC/2 - l_\infty)) = \bigoplus_{d=0}^{n/2-1} H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(2d)) \quad (3.41)$$

by induction. Here n is even in this case. Since $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d))$ is the space of homogeneous polynomials in z, w of degree d , it is equal to $\sum_{i=0}^d t_1^{-i} t_2^{-d+i}$ in the representation ring of T^2 . Thus we have

$$L_{\alpha, \beta}^{n>1} = \sum_{d=0}^{n/2-1} \sum_{i=0}^{2d} t_1^{-i} t_2^{-2d+i} = \sum_{\substack{i, j \geq 0, i+j \equiv 0 \pmod{2} \\ i+j \leq n-2}} t_1^{-i} t_2^{-j}. \quad (3.42)$$

Next we consider the case of $n < -1$. We use $0 \rightarrow \mathcal{O}((-n-2)C/2) \rightarrow \mathcal{O}(-nC/2) \rightarrow \mathcal{O}_C(-nC/2) \rightarrow 0$ to obtain

$$0 \rightarrow H^1(F_2, \mathcal{O}((-n-2)C/2 - l_\infty)) \rightarrow H^1(F_2, \mathcal{O}(-nC/2 - l_\infty)) \rightarrow H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(n)) \rightarrow 0. \quad (3.43)$$

Here we use $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(n)) = 0$ in this case. Starting with $H^1(F_2, \mathcal{O}((-n-2)C/2 - l_\infty))$ for $n = -2$, we obtain

$$H^1(F_2, \mathcal{O}(-nC/2 - l_\infty)) = \bigoplus_{d=1}^{-n/2} H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-2d)) \quad (3.44)$$

by induction. To use the Serre duality for $H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-2d))$, we give the canonical bundle $K_{\mathbf{P}^1} \cong t_1^{-1} t_2^{-1} \mathcal{O}_{\mathbf{P}^1}(-2)$. Using this, $H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-2d))$ is the dual of $t_1^{-1} t_2^{-1} H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(2d-2))$. Thus we have

$$L_{\alpha, \beta}^{n<-1} = t_1 t_2 \sum_{d=1}^{-n/2} \sum_{i=0}^{2d-2} t_1^i t_2^{2d-2-i} = \sum_{\substack{i, j \geq 0, i+j \equiv 0 \pmod{2} \\ i+j \leq -n-2}} t_1^{i+1} t_2^{j+1}. \quad (3.45)$$

Now we move to the remaining three terms in (3.39). We have

$$\begin{aligned}
& - \sum_{i=0} (-1)^i \text{Ext}^i(\mathcal{O}(k_\alpha C/2), \mathcal{O}_{Z_\beta}(k_\beta C/2 - l_\infty)) - \sum_{i=0} (-1)^i \text{Ext}^i(\mathcal{O}_{Z_\alpha}(k_\alpha C/2), \mathcal{O}(k_\beta C/2 - l_\infty)) \\
& + \sum_{i=0} (-1)^i \text{Ext}^i(\mathcal{O}_{Z_\alpha}(k_\alpha C/2), \mathcal{O}_{Z_\beta}(k_\beta C/2 - l_\infty)) \\
= & - \sum_{i=0} (-1)^i \text{Ext}^i(\mathcal{O}, \mathcal{O}_{Z_\beta}((k_\beta - k_\alpha)C/2 - l_\infty)) - \sum_{i=0} (-1)^i \text{Ext}^i(\mathcal{O}_{Z_\alpha}((k_\alpha - k_\beta)C/2), \mathcal{O}(-l_\infty)) \\
& + \sum_{i=0} (-1)^i \text{Ext}^i(\mathcal{O}_{Z_\alpha}, \mathcal{O}_{Z_\beta}((k_\beta - k_\alpha)C/2 - l_\infty)). \tag{3.46}
\end{aligned}$$

Since we have a decomposition $Z_\alpha = Z_\alpha^1 \cup Z_\alpha^2$, we obtain the remaining terms in (3.46) as the direct sum of the corresponding terms for $Z_\alpha^1(Z_\beta^1)$ and $Z_\alpha^2(Z_\beta^2)$. This is because mixed terms such as $\sum_{i=0} \text{Ext}^i(\mathcal{O}_{Z_\alpha^1}(k_\alpha C/2), \mathcal{O}_{Z_\beta^2}(k_\beta C/2 - l_\infty))$ are zero.

First we consider the terms for $Z_\alpha^1(Z_\beta^1)$. We take a coordinate system $(x, y) = (z_1/z_0, w/z)$, which transforms $(t_1^2 x, t_2/t_1 y)$ under T^2 -action. Since the divisor C is given by $x = 0$, the multiplication by x^m induces an isomorphism $\mathcal{O}_{Z_\alpha^1}(mC) \cong \mathcal{O}_{Z_\alpha^1}$ of sheaves for $m \in \mathbf{Z}/2$. For an isomorphism of equivalent sheaves, we twist it as $\mathcal{O}_{Z_\alpha^1}(mC) \cong t_1^{2m} \mathcal{O}_{Z_\alpha^1}$. Hence we sum up (3.46) for p_1 and obtain

$$t_1^{k_\beta - k_\alpha} \left(- \sum_{i=0} (-1)^i \text{Ext}^i(\mathcal{O}, \mathcal{O}_{Z_\beta^1}(-l_\infty)) - \sum_{i=0} (-1)^i \text{Ext}^i(\mathcal{O}_{Z_\alpha^1}, \mathcal{O}(-l_\infty)) + \sum_{i=0} (-1)^i \text{Ext}^i(\mathcal{O}_{Z_\alpha^1}, \mathcal{O}_{Z_\beta^1}(-l_\infty)) \right) \tag{3.47}$$

Since Z_α^1 is supported at the single point p_1 , we can consider it as a subscheme of \mathbf{P}^2 supported at the origin $[1 : 0 : 0]$, where T^2 -action on \mathbf{P}^2 is $[z_0 : z_1 : z_2] \mapsto [z_0 : t_1^2 z_1 : t_2/t_1 z_2]$. Let I_α^1 be the corresponding ideal sheaves of $\mathcal{O}_{\mathbf{P}^2}$. Using the $0 \rightarrow I_\alpha^1 \rightarrow \mathcal{O}_{\mathbf{P}^2} \rightarrow \mathcal{O}_{\mathbf{P}^2}/I_\alpha^1 = \mathcal{O}_{Z_\alpha^1} \rightarrow 0$, we obtain

$$t_1^{k_\beta - k_\alpha} \left(\sum_{i=0} (-1)^i \text{Ext}^i(I_\alpha^1, I_\beta^1) - \sum_{i=0} (-1)^i \text{Ext}^i(\mathcal{O}_{\mathbf{P}^2}, \mathcal{O}_{\mathbf{P}^2}(-l_\infty)) \right). \tag{3.48}$$

The second term $\sum_{i=0} (-1)^i \text{Ext}^i(\mathcal{O}_{\mathbf{P}^2}, \mathcal{O}_{\mathbf{P}^2}(-l_\infty))$ is zero. Thus we can use Theorem 5 after replacing (t_1, t_2) by $(t_1^2, t_2/t_1)$, and obtain $N_\alpha^{\bar{Y}^1}(t_1^2, t_2/t_1)$.

The terms for $Z_\alpha^2(Z_\beta^2)$ can be calculated in the similar way. \square

The naive estimation of the tangent space $T_{(E, \Phi)} \hat{M}(2, k, n)$ of $\hat{M}(2, k, n)$ for odd k is done. The result is given in Sec.C. In this estimation, we neglect the obstruction coming from $H^1(F_2, \mathcal{O}(\pm C/2 - l_\infty)) \neq 0$. We have to estimate the tangent space $T_{(E, \Phi)} \hat{M}(2, 2m+1, n)$ precisely by considering the contribution from $H^1(F_2, \mathcal{O}(\pm C/2 - l_\infty)) \neq 0$. After estimating the tangent space $T_{(E, \Phi)} \hat{M}(2, 2m+1, n)$, we can derive Poincaré polynomials of $\hat{M}(2, 2m+1, n)$. Furthermore, since the tangent space $T_{(E, \Phi)} \hat{M}(r, k, n)$ is decomposed as (3.38), we can estimate the tangent space $T_{(E, \Phi)} \hat{M}(r, k, n)$ for general r, k , by using the data of $T_{(E, \Phi)} \hat{M}(2, 2m, n)$ and $T_{(E, \Phi)} \hat{M}(2, 2m+1, n)$. Therefore we can also derive Poincaré polynomials of $\hat{M}(r, k, n)$ for general r, k straightforwardly.

3.3 Betti Numbers of $\hat{M}(2, 2m, n)$

Using the results of the previous subsection, we can calculate Betti numbers of $M(r, n)(\hat{M}(2, 2m, n))$. Furthermore we obtain the generating function of them as a beautiful closed formula. An algorithm of calculation of Betti numbers of $M(r, n)$ is as follows[25]. For this purpose, we introduce one parameter subgroup $\lambda : \mathbf{C}^* \rightarrow \tilde{T}$ as

$$\lambda(t) = (t^{m_1}, t^{m_2}, t^{n_1}, \dots, t^{n_r}). \quad (3.49)$$

If we choose weights m_1, m_2, n_α generic, the Zariski closure of $\lambda(\mathbf{C}^*)$ is equal to the whole \tilde{T} , and the fixed point set of $\lambda(\mathbf{C}^*)$ coincides with that of \tilde{T} . For weights m_1, m_2, n_α , which satisfy

$$m_2 \gg n_1 > n_2 > \dots > n_r \gg m_1 > 0, \quad (3.50)$$

the fixed points satisfy $M(r, n)^{\lambda(\mathbf{C}^*)} = M(r, n)^{\tilde{T}}$. We denote a fixed point by $(E, \Phi) \in M(r, n)^{\tilde{T}}$. The tangent space at the fixed point $T_{(E, \Phi)}M(r, n)$ has a \tilde{T} -module structure and an induced \mathbf{C}^* -structure via λ satisfying (3.50). Using Theorem 5, the index D of the fixed point is given by twice the sum of the numbers of $s \in Y_{\alpha, \beta}$ for which the weight for \mathbf{C}^* is negative [23]. Considering the fixed points satisfying the constraint $|\vec{Y}| = n$, we obtain Poincaré polynomial $P_z(M(r, n))$ of $M(r, n)$ by summing up z^D . Poincaré polynomial of $\hat{M}(2, 2m, n)$ is also obtained in the similar way. However we consider another algorithm of calculation of Betti numbers of $\hat{M}(2, 2m, n)$ by choosing a special case[25]:

$$m_1 = m_2 \gg n_1 > n_2 > 0, \quad (3.51)$$

where m_1, n_α are generic. Since λ is not generic, the fixed points are different from those of \tilde{T} . We also denote these points by $(E, \Phi) = (I_1(k_1 C/2), \Phi_1) \oplus (I_2(k_2 C/2), \Phi_2)$, but they satisfy the following conditions:

- (1) $I_\alpha(k_\alpha/2)$ is the tensor product $I_\alpha \otimes \mathcal{O}(k_\alpha C/2)$, where $k_\alpha \in \mathbf{Z}$ and I_α is an ideal sheaf of 0-dimensional subscheme Z_α contained in $\widehat{\mathbf{C}^2/\mathbf{Z}_2} = F_2 \setminus l_\infty$.
- (2) Φ_α is an isomorphism from $(I_\alpha)_{l_\infty}$ to the α th factor of $\mathcal{O}_{l_\infty}^{\oplus r}$.
- (3) I_α is fixed by the diagonal subgroup $\Delta \mathbf{C}^*$ of $\mathbf{C}^* \times \mathbf{C}^*$, coming from that on F_2 .

The difference between (3.50) and (3.51) appears in these fixed points, but the sum of each contribution is the same. A typical difference can be seen in rank one case. The Hilbert scheme of point on surface appears in (3.50) case, while the Hilbert scheme of point on cotangent bundle of Riemann surface appear in (3.51) case[23]. The components of the fixed point set are parametrized by $(\vec{k}, \vec{Y}) = ((k_1, Y_1), (k_2, Y_2))$, whose constraint is

$$\sum_{\alpha} k_{\alpha} = k, \quad |\vec{Y}| + \frac{1}{r} \left| \frac{k_1}{2} - \frac{k_2}{2} \right|^2 = n. \quad (3.52)$$

Note that \vec{Y}^1 in (3.35) are set to be zero due to the property of I_α . Now we see the property of I_α in detail. A general point in the component (\vec{k}, \vec{Y}) is $(E, \Phi) = (I_1(k_1 C/2), \Phi_1) \oplus (I_2(k_2 C/2), \Phi_2)$ such that

- (1) the support of I_α consists of $P_1, P_2, \dots, P_{l(Y_\alpha)}$, contained in the exceptional curve C ,
- (2) if ξ is the inhomogeneous coordinate of $C = \mathbf{P}^1$ and η is the coordinate of the fiber $\widehat{\mathbf{C}^2/\mathbf{Z}_2} \cong \mathcal{O}(-2) \rightarrow C$,

$$I_\alpha = (\xi - \xi_1, \eta^{\lambda_1^\alpha}) \cap (\xi - \xi_2, \eta^{\lambda_2^\alpha}) \cap \dots \cap (\xi - \xi_{l(Y_\alpha)}, \eta^{\lambda_{l(Y_\alpha)}^\alpha}), \quad (3.53)$$

where $x_l = \xi(P_l)$.

The points $P_1, P_2, \dots, P_{l(Y_\alpha)}$ move in \mathbf{P}^1 , but their order is irrelevant when the values λ_l^α are the same. Thus the component is isomorphic to

$$S^{Y_1} \mathbf{P}^1 \times \dots \times S^{Y_r} \mathbf{P}^r. \quad (3.54)$$

We define $S^Y \mathbf{P}^1$ in the following way. For a Young diagram $Y = (\lambda_1 \geq \lambda_2 \geq \dots)$, we denote

$$Y = (1^{m_1} 2^{m_2} \dots), \quad (3.55)$$

where $m_i = \#\{l | \lambda_l = i\}$. Using this notation, $S^Y \mathbf{P}^1$ is given by

$$S^Y \mathbf{P}^1 = S^{m_1} \mathbf{P}^1 \times S^{m_2} \mathbf{P}^1 \times \dots = \mathbf{P}^{m_1} \times \mathbf{P}^{m_2} \times \dots. \quad (3.56)$$

Note that $S^m \mathbf{P}^1 = \mathbf{P}^m$.

Suppose $k \equiv 0 \pmod{2}$. Let (E, Φ) be a fixed point in the component corresponding to $((k_1, Y_1), (k_2, Y_2))$. Then, the tangent space $T_{(E, \Phi)} \hat{M}(2, 2m, n)$ is a $\Delta \mathbf{C}^* \times T^r$ -module. Using Theorem 6, we obtain

$$T_{(E, \Phi)} \hat{M}(2, 2m, n) = \sum_{\alpha, \beta} (L_{\alpha, \beta}^{\vec{k}}(t_1, t_1) + t^{k_\beta - k_\alpha} N_{\alpha, \beta}^{\vec{Y}}(1, t_1^2)). \quad (3.57)$$

Here we set $\vec{Y}^1 = 0$. Using this, we obtain

Theorem 7 *The poincaré polynomial of $\hat{M}(2, 2m, n)$ is given by*

$$P_t(\hat{M}(2, 2m, n)) = \sum_{\substack{|\vec{Y}| + \frac{1}{2} \left| \frac{k_1}{2} - \frac{k_2}{2} \right|^2 = n, \\ \sum_{\alpha} k_\alpha = 2m}} \prod_{\alpha=1}^2 t^{2(|Y_\alpha| - l(Y_\alpha))} P_t(S^{Y_\alpha} \mathbf{P}^1) \cdot t^{2(l' + |Y_\alpha| + |Y_\beta| - n')}, \quad (3.58)$$

where

$$l' = \begin{cases} (K + m)^2 & \text{if } k_\alpha = K + 2m \geq k_\beta = -K, \\ (-K - m)^2 - 1 & \text{if } k_\alpha = K + 2m < k_\beta = -K, \end{cases} \quad (3.59)$$

$$n' = \begin{cases} (\# \text{ of columns of } Y_\alpha \text{ which are longer than } K + m) & \text{if } K + m \geq 0, \\ (\# \text{ of columns of } Y_\beta \text{ which are longer than } -K - m - 1) & \text{if } K + m < 0. \end{cases} \quad (3.60)$$

The calculation for rank one is done in the same way. This implies that $P_t(\hat{M}(1, 0, n)) = P_t((\widehat{\mathbf{C}^2/\mathbf{Z}_2})^{[n]})$. On the other hand, under the action of (3.34) the fixed points $M_0(1, n)^{\tilde{T}}$ consist of the single point $n[0] \in S^n \mathbf{C}^2 \subset M_0(1, n)$. Thus, there is a projective morphism:

$$\hat{M}(1, 0, n) = (\widehat{\mathbf{C}^2/\mathbf{Z}_2})^{[n]} \rightarrow S^n \mathbf{C}^2 = M_0(1, n). \quad (3.61)$$

This is an evidence of Conjecture 1.

Concretely we give the generating function of the case of $m = 0$.

Theorem 1 *The generating function of Poincaré polynomial is*

$$\begin{aligned}
& \sum_{n \in \frac{\mathbf{Z}_{\geq 0}}{2}} P_t(\hat{M}(2, 0, n)) q^n \\
= & \prod_{d=1}^{\infty} \frac{1}{(1 - t^{4d} q^d)(1 - t^{4d-2} q^d)^2(1 - t^{4d-4} q^d)} \\
& \times \left[\sum_{k \geq 0} \prod_{d=1}^k \frac{1 - t^{4d-4} q^d}{1 - t^{4d} q^d} t^{2k^2} q^{\frac{k^2}{2}} + \sum_{k > 0} \prod_{d=1}^{k-1} \frac{1 - t^{4d-4} q^d}{1 - t^{4d} q^d} t^{2k^2-2} q^{\frac{k^2}{2}} \right] \quad (3.62)
\end{aligned}$$

$$= \prod_{d=1}^{\infty} \frac{1}{(1 - t^{4d} q^d)(1 - t^{4d-2} q^d)^2(1 - t^{4d-4} q^d)} \prod_{d=1}^{\infty} \frac{1 - (-t^2 q^{\frac{1}{2}})^d t^{-2}}{1 - (-t^2 q^{\frac{1}{2}})^d} \sum_{k \in \mathbf{Z}} t^{2k^2} q^{\frac{k^2}{2}} \quad (3.63)$$

$$= \prod_{d=1}^{\infty} \frac{1}{(1 - t^{4d} q^d)(1 - t^{4d-2} q^d)^2(1 - t^{4d-4} q^d)} \prod_{d=1}^{\infty} \frac{1 - (-t^2 q^{\frac{1}{2}})^d t^{-2}}{1 + (-t^2 q^{\frac{1}{2}})^d}. \quad (3.64)$$

The equality between (3.62) and (3.63) is verified in Sec.B. The middle factor in (3.63) is a typical difference from the twisted $\mathcal{N} = 4$ partition function. This factor causes the failure of describing the generating function by using level two affine Lie algebras[15](See also Sec.A). The second factor in (3.64) is corresponding to the contribution from vector bundles. We also calculate the generating function of Poincaré polynomial in the different way in the next section. There is also an identity.

Lemma 3

$$\sum_{n \in \frac{\mathbf{Z}_{\geq 0}}{2}} P_t(\hat{M}(2, 2m, n)) q^n = \sum_{n \in \frac{\mathbf{Z}_{\geq 0}}{2}} P_t(\hat{M}(2, 0, n)) q^n. \quad (3.65)$$

To obtain the full $U(2)$ generating function, we define

$$Z_{A_1}^{U(2)}(q, z) := \sum_{m, n \in \mathbf{Z}} P_t(\hat{M}(2, 2m, n)) q^{n + \frac{(2m)^2}{8}} z^{2m}. \quad (3.66)$$

By using Lemma 3 and Sec.A, this formula can be rewritten by

$$\begin{aligned}
Z_{A_1}^{U(2)}(q; z) &= \sum_{m, n \in \mathbf{Z}} P_t(\hat{M}(2, 2m, n)) q^{n + \frac{(2m)^2}{8}} z^{2m} \\
&= \sum_m q^{\frac{m^2}{2}} z^{2m} \sum_{n \in \frac{\mathbf{Z}_{\geq 0}}{2}} P_t(\hat{M}(2, 0, n)) q^n \\
&= \theta_3(q; z) \sum_{n \in \frac{\mathbf{Z}_{\geq 0}}{2}} P_t(\hat{M}(2, 0, n)) q^n \quad (3.67)
\end{aligned}$$

On the other hand, the $U(1)$ generating function of Poincaré polynomial on $\widehat{\mathbf{C}^2/\mathbf{Z}_2}$ was already calculated in [7, 8]. Following the method used in [7, 8],

we calculate the first several part of the $U(2)$ generating function of Poincaré polynomial and obtain the same result. Recently the closed formula of the $U(2)$ generating function of Poincaré polynomial was proposed by T.Hausel(See (9) in[11]). His result also coincides with the above result.

Finally we consider the parameters which satisfy

$$n_1 \gg n_2 \gg m_2 \gg m_1 > 0. \quad (3.68)$$

As mentioned in [7], this situation corresponds to the $U(1)^2$ case instead of the $U(2)$ case. The former corresponds to the Coulomb branch, while the latter corresponds to the Higgs branch. We denote the framed moduli space for $U(1)^2$ by $\hat{M}'(2, 2m, n)$ satisfying (3.35). Using Theorem 6, we obtain

Theorem 8

$$P_t(\hat{M}'(2, 0, n)) = \sum_{\substack{|\vec{Y}| + \frac{1}{2} \left| \frac{k_1}{2} - \frac{k_2}{2} \right|^2 = n, \\ \sum_{\alpha} k_{\alpha} = 0}} \prod_{\alpha=1}^2 t^{2(r|Y_{\alpha}^1| - l(Y_{\alpha}^1) + r|Y_{\alpha}^2|)} \cdot t^{2(\frac{k_{\alpha} - k_{\beta}}{2})^2}. \quad (3.69)$$

Furthermore we obtain

Lemma 4

$$\sum_{n \in \frac{\mathbb{Z}}{2}} P_t(\hat{M}'(2, 0, n)) q^n = \frac{\theta_3(t^4 q; 1)}{\prod_{d=1} (1 - t^{4d-2} q^d)^2 (1 - t^{4d} q^d)^2}. \quad (3.70)$$

This formula coincides with (3.64) under $t = -1$, which implies that the both formula is the same in the partition function level. We remark that the partition function is described as a level two A_1 theta function over eta functions up to q power. Following the full $U(2)$ partition function, we multiplying (3.70) with $t = -1$, by $\theta_3(q; 1)$ to obtain the same full $U(2)$ partition function as before. The resulting formula implies that the $U(2)$ partition function on $\widehat{\mathbf{C}^2/\mathbf{Z}_2}$ is the second power of the $U(1)$ partition function on $\mathbf{C}^2/\mathbf{Z}_2$. The first factor of the full $U(2)$ partition function comes from the untwisted sector, and the second factor comes from the twisted sector. Each factor is the same. This is a typical structure of the partition function on $\widehat{\mathbf{C}^2/\mathbf{Z}_2}$. The partition function on $\mathbf{C}^2/\mathbf{Z}_m$ must not have this structure in general.

4 Betti Numbers of Moduli Spaces of Rank Two Stable Sheaves on \widehat{X}

We give a universal relation between the generating function of Poincaré polynomial of the moduli spaces of rank two stable sheaves on X with an A_1 singularity at $p \in X$ and that on \widehat{X} blow-uped at p . We call this relation the $\mathcal{O}(-2)$ blow-up formula. In the derivation, we have a difficulty in the precise definition of stable vector bundles(sheaves) on a singular surface. Instead of discussing this difficult point, we consider the formal generalization of the $\mathcal{O}(-1)$ case [41] to the $\mathcal{O}(-2)$ case, and check this $\mathcal{O}(-2)$ blow-up formula in the concrete case of $X = \mathbf{C}^2/\mathbf{Z}_2$, by comparing the result in the previous section. Here, the moduli space of stable

vector bundles(sheaves) itself on $\mathbf{C}^2/\mathbf{Z}_2$ is derived by considering \mathbf{Z}_2 -invariant part of the moduli space of stable vector bundles(sheaves) on \mathbf{C}^2 . Main tools used in this section are Weil conjecture and elementary transformations. Thus all calculations are done over finite field. We separate our calculation into two parts: vector bundle part and non-vector bundle part. Since a stable sheaf E has a stable vector bundle $E^{\vee\vee}$ and E is naturally embedded into $E^{\vee\vee}$, there is an exact sequence

$$0 \rightarrow E \rightarrow E^{\vee\vee} \rightarrow E^{\vee\vee}/E \rightarrow 0. \quad (4.1)$$

Thus we count the numbers of stable vector bundle $E^{\vee\vee}$ and those of maps $E^{\vee\vee} \rightarrow E^{\vee\vee}/E$, which correspond to vector bundle part and non-vector bundle part respectively.

4.1 Moduli Space of Stable Vector Bundles

To consider the moduli space of stable vector bundles, let us remember stable torsion free sheaves E on \mathbf{P}^2 [31]. First we define $\mu(E)$ (resp. $p_E(k)$) in order to define a stable torsion free sheaf in the sense of Mumford and Takemoto[22, 37] (resp. in the sense of Gieseker and Maruyama[10, 20]): $\mu(E) := c_1(E)/rk(E)$ (rep. $p_E(k) := \chi(E \otimes \mathcal{O}_{\mathbf{P}^2}(k))/rk(E)$). We call E a stable torsion free sheaf in the sense of Mumford and Takemoto (in the sense of Gieseker and Maruyama) if $\mu(F) < \mu(E)$ ($p_F(k) < p_E(k)$ for $k \gg 0$) for all coherent subsheaves $F \subset E$ with $0 < rk F < rk E$. We also call E a semistable torsion free sheaf if $\mu(F) \leq \mu(E)$ ($p_F(k) \leq p_E(k)$ for $k \gg 0$) for all coherent subsheaves $F \subset E$ with $0 < rk F < rk E$. To distinguish a stable sheaf in the sense of Gieseker from that in the sense of Mumford, we call the latter μ -stable sheaf hereafter.

Let X be a surface over \mathbf{F}_q with an A_1 -singularity at $p \in X$ and H be an ample divisor on X . Suppose that a divisor L on X satisfies $(L, H) = \text{odd}$. We consider one point blowing-up of X at $p \in X$: $\phi : \hat{X} \rightarrow X$. Let C be the exceptional divisor with $C^2 = -2$. Then we pull back L by ϕ^* ,

$$\begin{array}{ccc} \phi^* : \text{Pic}(X) & \rightarrow & \text{Pic}(\hat{X}) \\ \downarrow & & \downarrow \\ L & \xrightarrow{\sim} & \phi^* L. \end{array} \quad (4.2)$$

Using the equivalence $L \cong \phi^* L$, we denote $\phi^* L$ by L again. Let E be a rank two vector bundle on \hat{X} with $c_1(E) = L + aC$, ($a = 0, -1/2, -1, -3/2$), $c_2(E) = n$.

If E is μ -stable with respect to $lH - C$ for sufficiently large $l \in \mathbf{Z}_{>0}$, we introduce these moduli spaces $M_{lH-C}(L + aC, n)_0 =: M_{H_\infty}(L + aC, n)_0$. Similarly we introduce the moduli space of μ -stable vector bundle with respect to H as $M_H(L + aC, n)_0$. We assume that $M_{H_\infty}(L + aC, n)_0$ and $M_H(L + aC, n)_0$ are compatible and deeply connected in the same way as the case of X = a smooth four surface and \hat{X} = the blow-up of X [2].

Definition 8 $\tilde{M}_{i,n}(d)$ is the set of vector bundles E such that

$$E|_C = \mathcal{O}_C(d + i) \oplus \mathcal{O}_C(-d) \quad (4.3)$$

and $E \in M_{H_\infty}(L - iC, n)_0(\mathbf{F}_q)$. Then $\#M_H(L - iC/2, n)_0 = \#\tilde{M}_{i,n}(0)$, $i = 0, 1$, $\#M_H(L - iC/2, n)_0 = \#\tilde{M}_{i,n}(-1)$, $i = 2, 3$.

First we consider the case of $a = 0, 2$. For an element E of $M_{H_\infty}(L, n)_0(\mathbf{F}_q)$ and a subjection $\varphi_d : E \rightarrow \mathcal{O}_C(d)$ with $d \in \mathbf{Z}$, we set $E' = \ker(\varphi_d)$. E' is an elementary transformation of E along $\mathcal{O}_C(d)$. E and E' are related by the following commutative diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \uparrow & & \uparrow & & \\
0 & \rightarrow & \mathcal{O}_C(-d) & \rightarrow & E|_C & \xrightarrow{\varphi_d|_C} & \mathcal{O}_C(d) \rightarrow 0 \\
& & \uparrow \varphi_{-d} & & \uparrow & & \uparrow \varphi_d \\
0 & \rightarrow & E' & \rightarrow & E & \rightarrow & \mathcal{O}_C(d) \rightarrow 0 \\
& & \uparrow & & \uparrow & & \parallel \\
& & E(-C) & = & E(-C) & & \\
& & \uparrow & & \uparrow & & \\
& & 0 & & 0 & &
\end{array} \quad (4.4)$$

We can easily obtain

$$c_1(E') = c_1(E) - C = L - C, \quad (4.5)$$

$$c_2(E') = c_2(E) + d = n + d. \quad (4.6)$$

We remark that the correspondence between (E, φ_d) and (E', φ_{-d}) is bijective.

To count vector bundles, we define the following set of maps:

Definition 9

$$\tilde{\Phi}_d(E) = \{\varphi | \varphi \text{ is a surjection } E \rightarrow \mathcal{O}_C(d)\} / \text{Aut}(E). \quad (4.7)$$

Definition 10 For $F = \mathcal{O}_C(d+c) \oplus \mathcal{O}_C(-d)$ with $c = 0, 1, 2, 3$,

$$\Phi_d(F) = \{\varphi | \varphi \text{ is a surjection } F \rightarrow \mathcal{O}_C(d)\} / \mathbf{F}_q^X, \quad (4.8)$$

where for $c = 0, 1$, $d \geq 0$ and for $c = 2, 3$, $d \geq -1$.

For the set, we have

Lemma 5 For $c = 0$,

$$\#\Phi_a(F) = \begin{cases} q+1, & \text{if } d = a = 0, \\ 1, & \text{if } d > 0, a = -d, \\ q^{2a+1}, & \text{if } d > 0, a = d, \\ q^{2a-1}(q^2-1), & \text{if } d \geq 0, a > d, \\ 0, & \text{otherwise.} \end{cases} \quad (4.9)$$

For $c = 1$,

$$\#\Phi_a(F) = \begin{cases} 1, & \text{if } d \geq 0, a = -d, \\ q^{2a}, & \text{if } a = d+1 \geq 1, \\ q^{2a-2}(q^2-1), & \text{if } a > d+1 \geq 1, \\ 0, & \text{otherwise.} \end{cases} \quad (4.10)$$

For $c = 2$,

$$\#\Phi_a(F) = \begin{cases} 1, & \text{if } d \geq 0, a = -d, \\ q^{2a-1}, & \text{if } a = d+2 \geq 2, \\ q^{2a-3}(q^2-1), & \text{if } a > d+2 \geq 1, \\ q+1, & \text{if } a = -d = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (4.11)$$

For $c = 3$,

$$\#\Phi_a(F) = \begin{cases} 1, & \text{if } d \geq -1, a = -d, \\ q^{2a-2}, & \text{if } a = d+3 \geq 2, \\ q^{2a-4}(q^2-1), & \text{if } a > d+3 \geq 2, \\ 0, & \text{otherwise.} \end{cases} \quad (4.12)$$

For $c = 0, 1$, Lemma 6 was already given in Lemma 1.5. in [41].

Since $\text{Aut}(E) = \mathbf{F}_q^X$, $\tilde{\Phi}_d(E) = \Phi_d(E|_C)$. By the above diagram, we obtain

$$\sum_{E \in M_{H_\infty}(L-C, n)} \#\tilde{\Phi}_{-d}(E) = \sum_{E \in M_{H_\infty}(L, n-d)} \#\tilde{\Phi}_d(E), \quad (d \geq 0), \quad (4.13)$$

$$\sum_{E \in M_{H_\infty}(L, n)} \#\tilde{\Phi}_{-d}(E) = \sum_{E \in M_{H_\infty}(L-C, n-d)} \#\tilde{\Phi}_d(E), \quad (d \geq 1). \quad (4.14)$$

Using $M_{H_\infty}(L, n)_0(\mathbf{F}_q) = \coprod_{l=0}^n \tilde{M}_{0,n}(l)$ and $M_{H_\infty}(L-C, n)_0(\mathbf{F}_q) = \coprod_{l=-1}^n \tilde{M}_{2,n}(l)$, we rewrite the above relations as follows:

Lemma 6

$$\#\tilde{M}_{0,n}(d) = \sum_{l=-1}^{d-3} q^{2d-3}(q^2-1)\#\tilde{M}_{2,n-d}(l) + q^{2d-1}\#\tilde{M}_{2,n-d}(d-2), \quad (d \geq 2), \quad (4.15)$$

$$\#\tilde{M}_{2,n}(d) = \sum_{l=0}^{d-1} q^{2d-1}(q^2-1)\#\tilde{M}_{0,n-d}(l) + q^{2d+1}\#\tilde{M}_{0,n-d}(d), \quad (d \geq 1), \quad (4.16)$$

$$\#\tilde{M}_{0,n}(1) = (q+1)\#\tilde{M}_{2,n-1}(-1), \quad (4.17)$$

$$\#\tilde{M}_{2,n}(0) = (q+1)\#\tilde{M}_{0,n}(0). \quad (4.18)$$

By using these lemmas, we find

$$\#M_{H_\infty}(L, n)_0(\mathbf{F}_q) = \sum_{k=0}^n B_k^0(q)\#\tilde{M}_{0,n-k}(0) + \sum_{k=0}^n B_k^2(q)\#\tilde{M}_{2,n-k-1}(-1), \quad (4.19)$$

$$\#M_{H_\infty}(L-C, n)_0(\mathbf{F}_q) = \sum_{k=0}^n B_k^2(q)\#\tilde{M}_{0,n-k}(0) + \sum_{k=0}^n B_k^0(q)\#\tilde{M}_{2,n-k}(-1). \quad (4.20)$$

We can arrange these formulas in the following form

$$\begin{aligned} & \sum_n (\#M_{H_\infty}(L, n)_0(\mathbf{F}_q) + t^{1/2}\#M_{H_\infty}(L-C, n)_0(\mathbf{F}_q))t^n \\ = & \left(\sum_n (B_n^0(q) + t^{1/2}B_n^2(q))t^n \right) \left(\sum_n (\#M_H(L, n)_0(\mathbf{F}_q) + t^{1/2}\#M_H(L-C, n)_0(\mathbf{F}_q))t^n \right). \end{aligned} \quad (4.21)$$

$\sum_n (B_n^0(q) + t^{1/2}B_n^2(q))t^n$ itself is given by

Theorem 9

$$\sum_n (B_n^0(q) + t^{1/2}B_n^2(q))t^n = \prod_{d \geq 1} \frac{1 - q^{d-1}(-t^{1/2})^d}{1 + q^d(-t^{1/2})^d}. \quad (4.22)$$

Next we consider the case of $a = 1, 3$. For an element E of $M_{H_\infty}(L - C/2, n)_0(\mathbf{F}_q)$ and a surjection $\varphi_{d+1} : E \rightarrow \mathcal{O}_C(d+1)$ with $d \in \mathbf{Z}$, we set $E' = \ker(\varphi_{d+1})$. E' is an elementary transformation of E along $\mathcal{O}_C(d+1)$. E and E' are related by the following commutative diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \uparrow & & \uparrow & & \\
0 & \rightarrow & \mathcal{O}_C(-d) & \rightarrow & E|_C & \xrightarrow{\varphi_{d+1}|_C} & \mathcal{O}_C(d+1) \rightarrow 0 \\
& & \uparrow \varphi_{-d} & & \uparrow & & \parallel \\
0 & \rightarrow & E' & \rightarrow & E & \xrightarrow{\varphi_{d+1}} & \mathcal{O}_C(d+1) \rightarrow 0 \\
& & \uparrow & & \uparrow & & \\
& & E(-C) & = & E(-C) & & \\
& & \uparrow & & \uparrow & & \\
& & 0 & & 0 & &
\end{array} \quad (4.23)$$

We can easily obtain

$$c_1(E') = c_1(E) - C = L - 3C/2, \quad (4.24)$$

$$c_2(E') = c_2(E) + d = n + d. \quad (4.25)$$

We remark that the correspondence between (E, φ_{d+1}) and (E', φ_{-d}) is bijective. By using the above diagram, we have

$$\sum_{E \in M_{H_\infty}(L-3C/2, n)} \#\tilde{\Phi}_{-d}(E) = \sum_{E \in M_{H_\infty}(L-C/2, n-d)} \#\tilde{\Phi}_{d+1}(E), \quad (d \geq 0), \quad (4.26)$$

$$\sum_{E \in M_{H_\infty}(L-C/2, n)} \#\tilde{\Phi}_{-d}(E) = \sum_{E \in M_{H_\infty}(L-3C/2, n-d-1)} \#\tilde{\Phi}_{d+1}(E), \quad (d \geq 0). \quad (4.27)$$

Using $M_{H_\infty}(L - C/2, n)_0(\mathbf{F}_q) = \coprod_{l=0}^n \tilde{M}_{1,n}(l)$ and $M_{H_\infty}(L - 3C/2, n)_0(\mathbf{F}_q) = \coprod_{l=-1}^n \tilde{M}_{3,n}(l)$, we rewrite the above relations as follows:

Lemma 7

$$\#\tilde{M}_{1,n}(d) = \sum_{l=-1}^{d-3} q^{2d-2}(q^2 - 1)\#\tilde{M}_{3,n-d-1}(l) + q^{2d}\#\tilde{M}_{3,n-d-1}(d-2), \quad (d \geq 1), \quad (4.28)$$

$$\#\tilde{M}_{3,n}(d) = \sum_{l=0}^{d-1} q^{2d}(q^2 - 1)\#\tilde{M}_{1,n-d}(l) + q^{2d+2}\#\tilde{M}_{1,n-d}(d), \quad (d \geq 0), \quad (4.29)$$

$$\#\tilde{M}_{1,n}(0) = \#\tilde{M}_{3,n-1}(-1). \quad (4.30)$$

By using these lemmas, we find

$$\#M_{H_\infty}(L - C/2, n)_0(\mathbf{F}_q) = \sum_{k=0}^n B_k^1(q)\#\tilde{M}_{1,n-k}(0), \quad (4.31)$$

$$\#M_{H_\infty}(L - 3C/2, n)_0(\mathbf{F}_q) = \sum_{k=0}^{n+1} B_k^1(q)\#\tilde{M}_{1,n-k+1}(0). \quad (4.32)$$

We can arrange these formulas in the following form

$$\begin{aligned} & \sum_n (\#M_{H_\infty}(L - C/2, n)_0(\mathbf{F}_q) + t\#M_{H_\infty}(L - 3C/2, n)_0(\mathbf{F}_q))t^n \\ &= 2\left(\sum_n B_n^1(q)t^n\right)\left(\sum_n \#M_H(L - C/2, n)_0(\mathbf{F}_q)t^n\right). \end{aligned} \quad (4.33)$$

$\sum_n B_n^1(q)t^n$ itself is given by

Theorem 10

$$\sum_n B_n^1(q)t^n = \prod_{d \geq 1} \frac{1 - q^{4d-2}t^{2d}}{1 - q^{4d-2}t^{2d-1}}. \quad (4.34)$$

4.2 O(-2) Blow-up Formula

To count the \mathbf{F}_q -rational points of the moduli space $M_{H_\infty}(L - iC/2, n)$, we have to consider stable sheaves of non-vector bundle (See Sec.5 in [41]). Let E be a stable sheaf of rank 2 on \hat{X} with $c_1(E) = L - iC/2, c_2(E) = n$. Then $E^{\vee\vee}$ is the double dual of E and a stable vector bundle. As mentioned above, there is an exact sequence:

$$0 \rightarrow E \rightarrow E^{\vee\vee} \rightarrow E^{\vee\vee}/E \rightarrow 0, \quad (4.35)$$

where $c_2(E^{\vee\vee}) = c_2(E) - n', n' = \dim_{\mathbf{F}_q} H^0(E^{\vee\vee}/E)$. The number of stable sheaves with the above conditions is $\#\text{Quot}_{E^{\vee\vee}/\hat{X}/\mathbf{F}_q}^{n'}(\mathbf{F}_q)$, where the Quot-scheme $\text{Quot}_{E^{\vee\vee}/\hat{X}/\mathbf{F}_q}^{n'}$ is the scheme parametrizing all quotients $\psi : E^{\vee\vee} \rightarrow A$ such that the Hilbert polynomial of A is n . Therefore $\#M_{H_\infty}(L - iC/2, n)(\mathbf{F}_q) = \sum_{j=0}^n \#\text{Quot}_{E^{\vee\vee}/\hat{X}/\mathbf{F}_q}^j(\mathbf{F}_q) \#M_{H_\infty}(L - iC/2, n - j)_0(\mathbf{F}_q)$, and

$$\sum_n \#M_{H_\infty}(L - iC/2, n)(\mathbf{F}_q)t^n = \left(\sum_n \#\text{Quot}_{\mathcal{O}_{\hat{X}}^{\oplus 2}/\hat{X}/\mathbf{F}_q}^n(\mathbf{F}_q) \right) \left(\sum_n \#M_{H_\infty}(L - iC/2, n)_0(\mathbf{F}_q) \right). \quad (4.36)$$

Here we use $E^{\vee\vee} \cong \mathcal{O}_{\hat{X}}^{\oplus 2}$. This formula can be interpreted as the relation between the uncompactified moduli spaces and the Gieseker moduli spaces [18, 19]. We quote the following useful formula.

Theorem 11 (Yoshioka) [41]

$$\sum_{n \geq 0} \#\text{Quot}_{\mathcal{O}_{\hat{X}}^{\oplus r}/X/\mathbf{F}_q}^n(\mathbf{F}_q) = \prod_{a \geq 1} \prod_{b=1}^r Z_q(X, q^{ra-b}t^a). \quad (4.37)$$

Since $\#\hat{X}(\mathbf{F}_q) = \#X(\mathbf{F}_q) + q$, we have

$$Z_q(\hat{X}, t) = Z_q(X, t) \times \frac{1}{1 - qt}. \quad (4.38)$$

Substituting (4.38) and (4.21) into (4.36), we obtain

$$\begin{aligned} & \sum_n (\#M_{H_\infty}(L, n)(\mathbf{F}_q) + t^{1/2}\#M_{H_\infty}(L - C, n)(\mathbf{F}_q))t^n \\ &= \frac{\sum_n (B_n^0(q) + t^{1/2}B_n^2(q))t^n}{\prod_{a=1} (1 - q^{2a-1}t^a)(1 - q^{2a}t^a)} \left(\sum_n (\#M_H(L, n) + t^{1/2}\#M_H(L - C, n))t^n \right). \end{aligned} \quad (4.39)$$

Similarly for (4.33), we have

$$\begin{aligned} & \sum_n (\#M_{H_\infty}(L - C/2, n)_0(\mathbf{F}_q) + t\#M_{H_\infty}(L - 3C/2, n)_0(\mathbf{F}_q))t^n \\ &= \frac{2 \sum_n B_n^1(q)t^n}{\prod_{a=1} (1 - q^{2a-1}t^a)(1 - q^{2a}t^a)} \left(\sum_n \#M_H(L - C/2, n)t^n \right). \end{aligned} \quad (4.40)$$

The second factor of (4.39)((4.40)) is the generating function of Poincaré polynomials of the moduli spaces of stable sheaves on X . We call this as the contribution from the untwisted sector. On the other hand, the first factor of (4.39)((4.40)) is the contribution from the twisted sector. This comes from the effect of the blow-up. Thus we call this the $\mathcal{O}(-2)$ blow-up formula.

4.3 $X = \mathbf{C}^2/\mathbf{Z}_2$ Case

Using (4.39), we derive the generating function of Betti numbers of the moduli space of semistable sheaves on $\mathbf{C}^2/\mathbf{Z}_2$ (Note that if $c_1 \equiv 0 \pmod{2}$, we must consider semistable sheaves[31]). For this purpose, we have to derive Betti numbers of the moduli space of semistable sheaves on $\mathbf{C}^2/\mathbf{Z}_2$. Considering that (4.36) is a universal relation, we apply (4.36) to $\mathbf{C}^2/\mathbf{Z}_2$ case, and separate this into the contribution from vector bundles and that from sheaves of non-vector bundle. To derive the contribution from vector bundles, we consult the relation $(\mathbf{C}^2/\Gamma)^{[n]} = ((\mathbf{C}^2)^{[n]})^\Gamma$ [23], where $(*)^\Gamma$ stands for Γ -invariant part of $(*)$. Although $((\mathbf{C}^2)^{[n]})^\Gamma = M(1, n)^\Gamma$ is the total contribution of rank 1 case, we use the notion that the moduli space of semistable vector bundles (sheaves) on \mathbf{C}^2/Γ can be obtained by taking Γ -invariant part of the moduli space on \mathbf{C}^2 . First we derive the moduli space of semistable vector bundles on \mathbf{C}^2 . Using Theorem 11, we obtain

$$\begin{aligned} \sum_{n \geq 0} \# \text{Quot}_{\mathcal{O}_{\mathbf{C}^2/\mathbf{F}_q}^{\oplus r}}^n(\mathbf{F}_q) &= \prod_{a \geq 1} \prod_{b=1}^r Z_q(\mathbf{C}^2, q^{ra-b}t^a) \\ &= \frac{1}{\prod_{a \geq 1} \prod_{b=1}^r (1 - q^{ra-b}t^a)}. \end{aligned} \quad (4.41)$$

Remembering the result of the framed moduli space of instanton on S^4 (for an example, Corollary 3.10 in [25]) and comparing this formula with (4.36), we conclude

$$\sum_n \#M_H(0, n)_0(\mathbf{F}_q) = 1, \sum_n \#M_H(-C, n)_0(\mathbf{F}_q) = 0. \quad (4.42)$$

Thus, as the moduli space of semistable vector bundles on $\mathbf{C}^2/\mathbf{Z}_2$, we obtain

$$\sum_n \#M_H(0, n)_{\mathbf{Z}_2}(\mathbf{F}_q) = 1, \sum_n \#M_H(-C, n)_{\mathbf{Z}_2}(\mathbf{F}_q) = 0, \quad (4.43)$$

where $M_H(0, 0)_{\mathbf{Z}_2} \ni \mathcal{O}_{(\mathbf{C}^2 - \{\bar{0}\})/\mathbf{Z}_2}^{\oplus r}$ comes from $M_H(0, 0)_0 \ni \mathcal{O}_{\mathbf{C}^2}^{\oplus r}$. Finally we obtain

$$\sum_n (\#M_{H_\infty}(0, n)(\mathbf{F}_q) + t^{1/2} \#M_{H_\infty}(-C, n)(\mathbf{F}_q))t^n$$

$$= \frac{\sum_n (B_n^0(q) + t^{1/2} B_n^2(q)) t^n}{\prod_{a=1} (1 - q^{2a-1} t^a) (1 - q^{2a} t^a)} \cdot \frac{1}{\prod_{a=1} (1 - q^{2a-2} t^a) (1 - q^{2a-1} t^a)}. \quad (4.44)$$

By replacing $t \rightarrow q, q \rightarrow t^2$, the right hand side is completely the same as the formula in the previous section. By replacing $c_2(E) = n \rightarrow c_2(E) - \frac{1}{4}c_1(E)^2$, the left hand side also coincides with that in the previous section. The equivalence between (3.64) and (4.44) implies that the $\mathcal{O}(-2)$ blow-up formula is the true one. Furthermore this implies that the calculation based on the fixed points of torus action is the same as the calculation based on the fixed points of Frobenius morphism.

5 Conclusion and Discussion

We derived the generating functions of Betti numbers of the framed moduli space of instantons on $\widehat{\mathbf{C}^2/\mathbf{Z}_2}$, under the assumption that the corresponding torsion free sheaves E have vanishing properties ($\text{Hom}(E, E(-l_\infty)) = \text{Ext}^2(E, E(-l_\infty)) = 0$). Combining Betti numbers of the moduli space of stable sheaves on $\mathbf{C}^2/\mathbf{Z}_2$ with this, we can obtain the $\mathcal{O}(-2)$ blow-up formula for rank two. However this formula comes only from the calculation on $\mathbf{C}^2/\mathbf{Z}_2$ and that on $\widehat{\mathbf{C}^2/\mathbf{Z}_2}$. It is not clear that the $\mathcal{O}(-2)$ blow-up formula is valid on any four surface with an A_1 -singularity.

On the other hand, we derived the $\mathcal{O}(-2)$ blow-up formula for rank two, by determining the relation between the generating function of Betti numbers of the moduli space of stable sheaves on \hat{X} and that on X . We also checked this $\mathcal{O}(-2)$ blow-up formula in $X = \mathbf{C}^2/\mathbf{Z}_2$ case. However the derivation was owing to the formal generalization of the $\mathcal{O}(-1)$ blow-up formula given by Yoshioka[41]. We did not consider the justification in $\mathcal{O}(-2)$ case in detail.

The above two methods to determine the $\mathcal{O}(-2)$ blow-up formula have both advantages and disadvantages. We assume that each method compensates each other. The resulting $\mathcal{O}(-2)$ blow-up formula must be the true one on any four surface with an A_1 -singularity. We also have to overcome the disadvantages in each side. Can we derive vanishing properties ($\text{Hom}(E, E(-l_\infty)) = \text{Ext}^2(E, E(-l_\infty)) = 0$) directly? Can we make mathematically more rigorous derivation for the $\mathcal{O}(-2)$ blow-up formula? The most difficult point is the treatment of a stable sheaf on a singular surface. One possibility to avoid this difficulty is considering the contribution from the untwisted sector instead of considering a stable sheaf on a singular surface. Moreover we must define fractional line bundles $\mathcal{O}(k_\alpha C/2)$ rigorously. Indirectly fractional line bundles are justified by the beautiful close formulas.

We want to estimate the tangent space $T_{(E, \Phi)} \hat{M}(2, 2m+1, n)$ soon. This completes the estimation of $T_{(E, \Phi)} \hat{M}(r, k, n)$ for general r, k , which is described by the data of $T_{(E, \Phi)} \hat{M}(2, 2m, n)$ and $T_{(E, \Phi)} \hat{M}(2, 2m+1, n)$. Then, we can derive Poincaré polynomials of $\hat{M}(r, k, n)$ by using data of the tangent space. According to the results of the $U(2)$ partition function on $\widehat{\mathbf{C}^2/\mathbf{Z}_2}$, we assume that the $U(r)$ partition function on $\mathbf{C}^2/\mathbf{Z}_2$ is also described by a level 2 theta function $\theta_{A_{r-1}}^{(2)}$. By considering the full $U(r)$ partition function, we would find the duality between the $U(r)$ partition function on $\widehat{\mathbf{C}^2/\mathbf{Z}_2}$ and the $U(2)$ partition

function on $\widehat{\mathbf{C}^2/\mathbf{Z}_r}$, which is concerned about the level-rank duality of WZW models in conformal field theory[28].

We can consider instanton calculus on $\widehat{\mathbf{C}^2/\mathbf{Z}_m}$. Then, we assume that the $U(n)$ partition function on $\widehat{\mathbf{C}^2/\mathbf{Z}_m}$ is also described by affine Lie algebras. We are interested in how the $U(n)$ partition function on $\widehat{\mathbf{C}^2/\mathbf{Z}_m}$ can be separated into the contribution from the twisted sector and that from the untwisted sector.

We can also consider instanton calculus on $\widehat{\mathbf{C}^2/\mathbf{Z}_2}$ for D, E gauge groups. Then, we can use the duality between the D, E partition function on $\widehat{\mathbf{C}^2/\mathbf{Z}_2}$ and the $U(2)$ partition function on $\widehat{\mathbf{C}^2/\Gamma_{D,E}}$. Furthermore we can derive the corresponding blow-up formula. These calculations must serve to a further insight into our works for D, E gauge groups[14, 34, 35, 36].

What is the meaning of the middle factor in (3.63) ? Since this factor does not come from affine Lie algebras, it must be complicated to derive the generating functions for D, E gauge groups.

We want to verify the equivalence between the generating function on orbifold T^4/\mathbf{Z}_2 and that on $K3$, by using the $\mathcal{O}(-2)$ blow-up formula derived in this article. Then, we are interested in how 16th power of the $\mathcal{O}(-2)$ blow-up formula are combined with the untwisted sector of the generating function on orbifold T^4/\mathbf{Z}_2 . We hope to verify the equivalence by not assuming the duality conjecture.

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A Theta Functions

A.1 Level l Theta Functions

Following [15], a level l A_r theta function for weight β is given by

$$\theta_{A_r}^{(\beta(l))}(\tau; y) := \sum_{m \in \mathbf{Z}^r} q^{\frac{l}{2} t(m + \frac{1}{l} A_r^{-1} \beta) A_r(m + \frac{1}{l} A_r^{-1} \beta)} e^{2\pi i l^t(m + \frac{1}{l} A_r^{-1} \beta) A_r y}, \quad {}^t\beta = (b_1, b_2, \dots, b_{N-1}), b_i \in \mathbf{Z}, y \in \mathbf{Z}^r. \quad (\text{a.1})$$

Here A_r stands for the Cartan matrix of A_r again. $q = \exp(2\pi i \tau)$.

Using this notation, we give Jacobi's theta functions:

$$\theta_3(\tau; y) = \theta_{A_1}^{0(2)}(\tau; y) + \theta_{A_1}^{2(2)}(\tau; y), \quad (\text{a.2})$$

$$\theta_4(\tau; y) = \theta_{A_1}^{0(2)}(\tau; y) - \theta_{A_1}^{2(2)}(\tau; y), \quad (\text{a.3})$$

$$\theta_2(\tau; y) = \theta_{A_1}^{1(2)}(\tau; y). \quad (\text{a.4})$$

Similarly we define $\theta_{A_r}^{(2)}(\tau; y)$ as

$$\theta_{A_r}^{(2)}(\tau; y) := \sum_{\substack{{}^t(A_r \beta) = (\epsilon_1, \epsilon_2, \dots, \epsilon_r) \\ \epsilon_k = 0 \text{ or } 1}} \theta_{A_r}^{\beta(2)}(\tau; y). \quad (\text{a.5})$$

Using $q = \exp(2\pi i)$ and $z = \exp(2\pi i y)$, we sometimes denote these theta functions by $\theta_{A_r}^{(2)}(q; y)$, $\theta_{A_r}^{(2)}(\tau; z)$ and $\theta_{A_r}^{(2)}(q; z)$. Note that $\theta_{A_r}^{(2)}(\tau; y = 0)$ is invariant under $\tau \rightarrow -\frac{1}{\tau}$ up to a factor.

A.2 Level 2 Theta Functions via Instanton Calculus on $\widehat{\mathbf{C}^2/\mathbf{Z}_2}$

First we derive the constraint (3.52) from $E = \sum_{\alpha} I_{\alpha} \otimes \mathcal{O}(k_{\alpha}C/2)$. We have $c_1(E) = \sum_{\alpha} k_{\alpha}C/2$, $c_2(E) = \sum_{\alpha} |Y_{\alpha}| + C^2 \sum_{\alpha < \beta} k_{\alpha}k_{\beta}/4$ at the fixed point. Then we obtain

$$\begin{aligned} c_2(E) - \frac{r-1}{2r}c_1(E)^2 &= \sum_{\alpha} |Y_{\alpha}| + C^2 \sum_{\alpha < \beta} \frac{k_{\alpha}k_{\beta}}{4} - \frac{r-1}{2r}C^2 \frac{(\sum_{\alpha} k_{\alpha})^2}{4} \\ &= \sum_{\alpha} |Y_{\alpha}| + \frac{1}{r} \sum_{\alpha < \beta} \left| \frac{k_{\alpha}}{2} - \frac{k_{\beta}}{2} \right|^2. \end{aligned} \quad (\text{a.6})$$

Under the condition $\sum_{\alpha} k_{\alpha} = k$, we show that $\sum_{k_{\alpha} \in \mathbf{Z}} q^{\frac{1}{r} \sum_{\alpha < \beta} |\frac{k_{\alpha}}{2} - \frac{k_{\beta}}{2}|^2}$ becomes level 2 theta functions. By reparametrizing $\sum_{\alpha=i}^r k_{\alpha} = K_i$, we obtain

$$\begin{aligned} \frac{1}{r} \sum_{\alpha < \beta} \left| \frac{k_{\alpha}}{2} - \frac{k_{\beta}}{2} \right|^2 &= -\frac{1}{2} \sum_{\alpha < \beta} k_{\alpha}k_{\beta} + \frac{r-1}{r}k^2 \\ &= \frac{1}{2}(K_2 - k)K_2 + \frac{1}{2} \sum_{i=3}^r (K_i - K_{i-1})K_i + \frac{r-1}{r}k^2 \\ &= {}^t \left(\frac{K}{2} - \frac{k}{2} A_{r-1}^{-1} v \right) A_{r-1} \left(\frac{K}{2} - \frac{k}{2} A_{r-1}^{-1} v \right), \end{aligned} \quad (\text{a.7})$$

where $K \in \mathbf{Z}^{r-1}$, ${}^t v = (1, 0, \dots, 0)$. This is nothing but the exponent of (a.1) for $l = 2$ case. Conversely if $k_{\alpha}/2$ is replaced by k_{α} in (a.7), we would not obtain the level 2 theta functions.

B The Proof of the Equality between (3.62) and (3.63)

Let

$$(a)_k = (a; q)_k = (1-a)(1-aq) \cdots (1-aq^{k-1}), (a)_{\infty} = \prod_{d=0}^{\infty} (1-aq^d). \quad (\text{b.1})$$

We start with the formula of Entry 7(p.16) in [33]. We substitute $a = b$, then we obtain

$$\sum_{k \geq 0} \frac{(d/c)_k (d/q)_k (1-dq^{2k-1}) c^k q^{k(k-1)}}{(c)_k (q)_k} = \frac{(d/q)_{\infty}}{(c)_{\infty}}. \quad (\text{b.2})$$

We put $c = -q$, $d = q^2 a$,

$$\sum_{k \geq 0} \frac{(-qa)_k (qa)_k (1-aq^{2k+1}) (-1)^k q^{k^2}}{(-q)_k (q)_k} = \frac{(qa)_{\infty}}{(-q)_{\infty}}. \quad (\text{b.3})$$

Using $(e)_k (-e)_k = (e^2; q^2)_k$,

$$\begin{aligned} &\sum_{k \geq 0} \left(\frac{(q^2 a^2; q^2)_k}{(q^2; q^2)_k} (-q)^{k^2} + \frac{(q^2 a^2; q^2)_k}{(q^2; q^2)_k} (-q)^{(k+1)^2} a \right) \\ &= \frac{(qa)_{\infty}}{(-q)_{\infty}}. \end{aligned} \quad (\text{b.4})$$

We set $q = -t^2 q^{\frac{1}{2}}, a = t^{-2}$,

$$\begin{aligned} & \sum_{k \geq 0} \left(\prod_{d=1}^k \frac{1 - t^{4d-4} q^d}{1 - q^{4d} q^d} t^{2k^2} q^{\frac{k^2}{2}} + \prod_{d=1}^{k-1} \frac{1 - t^{4d-4} q^d}{1 - q^{4d} q^d} t^{2k^2-2} q^{\frac{k^2}{2}} \right) \\ &= \prod_{d=1}^{\infty} \frac{1 - (-t^2 q^{\frac{1}{2}})^d t^{-2}}{1 + (-t^2 q^{\frac{1}{2}})^d}. \end{aligned} \quad (\text{b.5})$$

We verify the equality between (c.5) and (c.6). We put $c = q^{\frac{3}{2}}, d = aq^2$ for Entry 7, and obtain

$$\sum_{k \geq 0} \frac{(aq^{\frac{1}{2}})_k (aq)_k (1 - aq^{2k+1}) q^{k^2 + \frac{k}{2}}}{(q^{\frac{3}{2}})_k (q)_k} = \frac{(aq)_{\infty}}{(q^{\frac{3}{2}})_{\infty}}. \quad (\text{b.6})$$

Using $(e)_k (eq^{\frac{1}{2}})_k = (e; q^{\frac{1}{2}})_{2k}$,

$$\sum_{k \geq 0} \frac{(aq^{\frac{1}{2}}; q^{\frac{1}{2}})_{2k} (1 - aq^{2k+1}) q^{k^2 + \frac{k}{2}}}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2k} (1 - q^{k+\frac{1}{2}})} = \frac{(aq)_{\infty}}{(q^{\frac{1}{2}})_{\infty}}. \quad (\text{b.7})$$

$$\sum_{k \geq 0} \frac{(aq^{\frac{1}{2}}; q^{\frac{1}{2}})_{2k}}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2k}} q^{k^2 + \frac{k}{2}} \left(1 + \frac{1 - aq^{k+\frac{1}{2}}}{1 - q^{k+\frac{1}{2}}} q^{k+\frac{1}{2}} \right) = \frac{(aq)_{\infty}}{(q^{\frac{1}{2}})_{\infty}}. \quad (\text{b.8})$$

$$\sum_{k \geq 0} \left(\frac{(aq^{\frac{1}{2}}; q^{\frac{1}{2}})_{2k}}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2k}} q^{\frac{(2k)^2 + 2k}{4}} + \frac{(aq^{\frac{1}{2}}; q^{\frac{1}{2}})_{2k+1}}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2k+1}} q^{\frac{(2k+1)^2 + 2k+1}{4}} \right) = \frac{(aq)_{\infty}}{(q^{\frac{1}{2}})_{\infty}}. \quad (\text{b.9})$$

$$\sum_{k \geq 0} \frac{(aq^{\frac{1}{2}}; q^{\frac{1}{2}})_k}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_k} q^{\frac{k^2 + k}{4}} = \frac{(aq)_{\infty}}{(q^{\frac{1}{2}})_{\infty}}. \quad (\text{b.10})$$

We set $q = t^8 q^2, a = t^{-4}$,

$$\sum_{k \geq 0} \prod_{d=1}^k \frac{1 - t^{4d-4} q^d}{1 - t^{4d} q^d} t^{2k^2 + 2k} q^{\frac{k^2 + k}{2}} = \prod_{d=1}^{\infty} \frac{1 - t^{8d-4} q^{2d}}{1 - t^{8d-4} q^{2d-1}}. \quad (\text{b.11})$$

C Betti Numbers of $\hat{M}(r, 2m+1, n)$

As in the proof of the theorem 6, for odd n $H^1(F_2, \mathcal{O}(-nC/2 - l_{\infty}))$ is not equal to

$$\bigoplus_{d=0}^{(n-1)/2-1} H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(2d+1)). \quad (\text{c.1})$$

However by using this to count indices, we obtain the following formulas.

The poincaré polynomial of $\hat{M}(2, 2m+1, n)$ is given by

$$P_t(\hat{M}(2, 2m+1, n)) = \sum_{\alpha=1}^2 \prod_{\alpha=1}^2 t^{2(|Y_{\alpha}| - l(Y_{\alpha}))} P_t(S^{Y_{\alpha}} \mathbf{P}^1) \cdot t^{2(l' + |Y_1| + |Y_2| - n')}, \quad (\text{c.2})$$

where $k \equiv 1$ case,

$$l' = \begin{cases} (K+m)^2 + (K+m) & \text{if } k_1 = K+2m+1 > k_2 = -K, \\ (-K-m)^2 + (K+m) & \text{if } k_1 = K+2m+1 < k_2 = -K, \end{cases} \quad (\text{c.3})$$

$$n' = \begin{cases} (\# \text{ of columns of } Y_1 \text{ which are longer than } K + m) & \text{if } K + m \geq 0, \\ (\# \text{ of columns of } Y_2 \text{ which are longer than } -K - m - 1) & \text{if } K + m < 0. \end{cases} \quad (\text{c.4})$$

The generating function of Poincaré polynomial in the case of $r = 2, c_1 = 1$ is

$$\begin{aligned} & \sum_{n \in \mathbf{Z}_{\geq 0}} P_t(\hat{M}(2, 1, n + \frac{1}{8})) q^{n + \frac{1}{8}} \\ = & \prod_{d=1}^{\infty} \frac{1}{(1 - t^{4d} q^d)(1 - t^{4d-2} q^d)^2(1 - t^{4d-4} q^d)} \\ & \times \left[\sum_{k \geq 0} \prod_{d=1}^k \frac{1 - t^{4d-4} q^d}{1 - t^{4d} q^d} t^{2k^2+2k} q^{\frac{(k+\frac{1}{2})^2}{2}} + \sum_{k > 0} \prod_{d=1}^{k-1} \frac{1 - t^{4d-4} q^d}{1 - t^{4d} q^d} t^{2k^2-2k} q^{\frac{(k-\frac{1}{2})^2}{2}} \right] \end{aligned} \quad (\text{c.5})$$

$$= \prod_{d=1}^{\infty} \frac{1}{(1 - t^{4d} q^d)(1 - t^{4d-2} q^d)^2(1 - t^{4d-4} q^d)} \prod_{d=1}^{\infty} \frac{1 - t^{8d-4} q^{2d}}{1 - t^{8d} q^{2d}} \sum_{k \in \mathbf{Z}} t^{2k^2+2k} q^{\frac{(k+\frac{1}{2})^2}{2}} \quad (\text{c.6})$$

$$= 2q^{\frac{1}{8}} \prod_{d=1}^{\infty} \frac{1}{(1 - t^{4d} q^d)(1 - t^{4d-2} q^d)^2(1 - t^{4d-4} q^d)} \prod_{d=1}^{\infty} \frac{1 - t^{8d-4} q^{2d}}{1 - t^{8d-4} q^{2d-1}}. \quad (\text{c.7})$$

This result is consistent with (4.40).

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